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Quantum theory of gravitational collapse (lecture notes on quantum conchology)

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Abstract

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1 Introduction

The issue of gravitational collapse includes not only the theory of black holes and of their classical and quantum properties, but also the serious problem of singularities. We believe that the singularities can be cured by quantum theory, but not by any semiclassical approximation thereof. The classical solutions near the singularity has to be changed strongly. New ideas concerning singularities are likely to influence the theory of black holes, too.

The rejection of WKB approximations forces us to another kind of approximative schema. Simplified models constitute a promising alternative. Our method of dealing with the problem will, therefore, employ simplified models and a kind of effective theory of gravity. It does not worry about the final form of a full-fledged theory of quantum gravity. This need not be completely unreasonable approach. Even if the ultimate quantum gravity theory were known, most calculations would still be performed by an approximation schema within some effective theory and for simplified models, such as the method of effective quantum theory in QCD. Another example of the method of simplified models used successfully today is the theory of atomic spectra, which belongs, strictly speaking, to the field of quantum electrodynamics. Still, infinite number of degrees of freedom can be safely frozen and Schroedinger equation applied. Such methods can give useful hints for the gravitational collapse also because of the fact that the mainstay of the black hole geometry is formed by the purely dependent degrees of freedom of the gravitational field (uniqueness theorems!), and these degrees of freedom have no proper quantum character of their own.

The model we work with is a spherically symmetric thin shell. Historically, the first physicist to use shells in quantum theory seems to be Dirac [1]. We shall try to do justice to the model and to take it absolutely seriously. There is no point in rushing towards prejudiced conceptions disregarding all kinds of problems that may emerge on the way.

These lecture notes are to explain in a coherent way a number of ideas scattered in various papers but they also contain some new results. Sec. 2 gives a general account of some gauge-invariant methods in the canonical theory of generally covariant systems, whose spiritual fathers are Dirac, Bergmann, Kuchař and Rovelli. New seems to be the observation that many conceptual and practical problems admit a solution if one restrict oneself to asymptotically flat models. This is a natural framework for the gravitational collapse. The asymptotic structure defines a decomposition of the full diffeomorphism group into the gauge group, the group of asymptotic symmetries and the rest that can be ignored. A physically meaningful choice of a complete set of gauge-invariant quantities—the so-called Dirac observables—is also provided by the structure. The asymptotic symmetries determine a gauge-invariant dynamics.

In this way, some aspects of the notorious problem of time are dealt with.

Sec. 3 starts with an application of the methods to the model of a single, spherically symmetric, thin shell of light-like matter surrounded by its own gravitational field. The study of the space of the corresponding solutions to Einstein equations leads to the identification of a complete set of Dirac observables. Thus, the basic observables are found independently of the asymptotic structure in this (lucky) case, but they can still be considered as asymptotic properties of the system. The canonical theory of the shell due to Louko, Whiting and Friedman is utilized to calculate the Poisson brackets of the observables and to find the generators of the asymptotic symmetries. The main new point is a careful description of asymptotic frames.

Finally, a quantum theory of the model is constructed in Sec. 4. The method of choice is the group-theoretical quantization because the phase space has a complicated boundary and the group quantization has been invented to deal with such a situation. The results are rather surprising: Some quantum shells contract, cross their Schwarzschild radius inwards, bounce at the centre without creating a singularity, cross their Schwarzschild radius outwards, and re-expands to the infinity from which they originally came. This is no joke! It is possible because the quantum Schwarzschild radius is a mixture of black and white hole states. It is amusing that an old calculation [2] based on a very different technique has come to somewhat similar results. A couple of ideas are listed of how these results could be reconciled with the “observational evidence for black holes” in astrophysics. An interesting problem for future research arises: what is the nature of quantum geometry?

2 Gauge-invariant method in canonical theory of generally covariant systems

The present section attempts to give a general account of some new methods that have been developed in a couple of recent years. It is also a description of a project. It contains many claims without proofs. Those of these claims that make sense for finite dimensional systems have been proved and discussed extensively in [3] and [4] for general finite dimensional systems. The rest of the claims without proofs have hitherto been proved only for a few special cases, one of which will be described in the next section.

2.1 Space of solutions, gauge group and asymptotic symmetries

Mathematically, any *generally covariant system* is constituted by a set Ψ of geometrical objects (fields, and submanifolds such as particles or shells) on a background

manifold \mathcal{M} . The set Ψ always includes a Lorentzian metric g on \mathcal{M} ; it is sufficient to consider Ψ as consisting just of g in order to understand all manipulations with Ψ in this lecture. Details about complicated systems are explained in [5] and [6]. The background \mathcal{M} has just a topological and differential structure (bare manifold without boundary) and has the meaning of a *spacetime* manifold. It can be of any dimension, but we assume here that it is four-dimensional. Several different backgrounds can be needed for one generally covariant system. For example, all three-dimensional manifolds that admit a Riemannian metric can play a role of Cauchy surfaces in general relativity; different topologies of Cauchy surface define different background manifolds.

The object Ψ has to satisfy a dynamical equation on \mathcal{M} (such as Einstein equations for g). We allow for a domain \mathcal{D}_Ψ of Ψ in \mathcal{M} where the solution is well defined to be a proper subset of \mathcal{M} . General covariance is the following property. Let $\text{Diff}\mathcal{M}$ be the diffeomorphism group of \mathcal{M} and let (\mathcal{M}, Ψ) be a spacetime that solves the dynamical equation. Then $(\mathcal{M}, \varphi_*\Psi)$ is another solution for any $\varphi \in \text{Diff}\mathcal{M}$. The symbol φ_* denotes the push forward of all geometrical objects collected within Ψ by φ . For example, let \mathcal{M} admits a global coordinate chart and let $\{X^\mu\}$ denote some global coordinates on \mathcal{M} ; the action of a diffeomorphism φ on a metric field $g_{\mu\nu}(X)$ has the following result:

$$(\varphi_*g)_{\mu\nu}(X) = g_{\rho\sigma}(Y(X)) \frac{\partial Y^\rho}{\partial X^\mu} \frac{\partial Y^\sigma}{\partial X^\nu} ,$$

where the functions $Y^\rho(X)$ are defined by φ and define the action of φ in terms of the coordinates $\{X^\mu\}$:

$$Y^\rho(X) = X^\rho(\varphi^{-1}(X)) .$$

The map φ_* is an “active” transformation so that e.g. φ_*g is a different field on \mathcal{M} than g is, in general. If Ψ and φ are such that $\varphi_*\Psi = \Psi$, then we say that Ψ admits a *spacetime symmetry* φ . The reason why we prefer to work with active transformations rather than coordinate transformations is a technical one: active transformations can be globally defined even if there are no global coordinate charts. In consequence of that, they form groups so that the whole powerful apparatus of group theory can be applied.

We shall restrict ourselves to the special case of systems with *asymptotically flat* solutions. There are coordinate free methods (due to Penrose) to decide whether or not a spacetime is asymptotically flat (see, e.g., [7] or [8]). Then there is a universal *asymptotic boundary* $\partial_{\text{as}}\mathcal{M}$ carrying an *asymptotic structure* that can be considered as common to all such solutions. For example, asymptotically flat solutions to Einstein equations all have the same scri, which has a geometrical structure that is richer than just the topological and differential one. This structure posses a

symmetry group that we call G_{as} ¹. We shall not go into detail here, but we shall give a more complete account for an example later. The asymptotic structure enables to select *standard asymptotical frames of reference* at $\partial_{\text{as}}\mathcal{M}$ that are also common to all solutions and are inhabited by *asymptotic observers*.

One can try to give a more precise account of these ideas as follows. Let $\tilde{\Gamma}$ be the space of all asymptotically flat solutions on \mathcal{M} . We assume that there is a subspace $\tilde{\Gamma} \subset \bar{\Gamma}$ and a subgroup G of $\text{Diff}\mathcal{M}$ with the following properties.

1. For each $\Psi \in \tilde{\Gamma}$, the spacetime (\mathcal{M}, Ψ) can be conformally extended to $(\bar{\mathcal{M}}_\Psi, \Psi)$ by attaching a scri \mathcal{I}_Ψ to \mathcal{M} and a neighbourhood $\tilde{\mathcal{I}}_\Psi$ of \mathcal{I}_Ψ in $\bar{\mathcal{M}}_\Psi$. Then there is a diffeomorphism $\Phi_\Psi : \tilde{\mathcal{I}}_\Psi \mapsto \bar{\mathcal{M}}_0$, such that Φ_Ψ maps \mathcal{I}_Ψ onto \mathcal{I} , where $\bar{\mathcal{M}}_0$ is the conformal compactification of Minkowski spacetime and \mathcal{I} is the scri of $\bar{\mathcal{M}}_0$.
2. Let $\varphi \in G$, $\Psi \in \tilde{\Gamma}$, $\Psi' = \varphi_*\Psi$ and $U = \Phi_\Psi\tilde{\mathcal{I}}_\Psi \cap \Phi_{\Psi'}\tilde{\mathcal{I}}_{\Psi'}$. Then φ can be differentiably extended to \mathcal{I}_Ψ and $\varphi_\Psi : U \mapsto U$ defined by $\varphi_\Psi = \Phi_{\Psi'} \circ \varphi \circ \Phi_\Psi^{-1}$ induces a map $\varphi_\Psi|_{\mathcal{I}}$ on \mathcal{I} of Minkowski spacetime that preserves the asymptotic structure and is an element of the asymptotic symmetry group G_{as} , $\varphi_\Psi|_{\mathcal{I}} \in G_{\text{as}}$.
3. For given $\varphi \in G$, $\varphi_\Psi|_{\mathcal{I}}$ is independent of Ψ . In this way, there is a well-defined map $\pi_{\text{as}} : G \mapsto G_{\text{as}}$, and we assume that π_{as} is onto. It follows that this map is a homeomorphism if it exists. Let us define the *gauge group* G_0 of the system to be the kernel of π_{as} . It follows that G_0 is a normal subgroup of G .
4. For each $\varphi \in G_0$ and $\Psi \in \tilde{\Gamma}$, the extension of φ to \mathcal{I}_Ψ is an odd-parity supertranslation² on \mathcal{I}_Ψ .

If this procedure works, then it is surely non unique. For example, \mathcal{I} can be chosen to lie in a “different corner” of \mathcal{M} . We suppose that the non-uniqueness can either be limited by some suitable additional requirements or that it will not manifest itself in physical results.

The quotient $\Gamma := \tilde{\Gamma}/G_0$ is the *physical phase space*. (To be a full-fledged phase space, it has to be equipped with a symplectic structure.) Let us denote the projection of the quotient by $\pi_{\text{sol}} : \tilde{\Gamma} \mapsto \tilde{\Gamma}/G_0$.

Let $\varphi_1 \in G \setminus G_0$. Consider the action of φ_1 on a point Ψ_1 of an orbit γ of G_0 in $\tilde{\Gamma}$. Clearly, $\varphi_{1*}\Psi$ lies in another orbit γ' and $\gamma' \neq \gamma$. Let $\Psi_2 \in \gamma$ and $\Psi_2 \neq \Psi_1$. Does $\varphi_{1*}\Psi_2$ lie in the same orbit as $\varphi_{1*}\Psi_1$? It does because G_0 is a normal subgroup of G . Hence φ_1 sends orbits into orbits. Further, let φ_2 be from the same class of G/G_0

¹For example, a symmetry of the geometrical structure of flat spacetime is an isometry of the spacetime; the symmetry group of it is the Poincaré group.

²Justification for this assumption must be taken from the canonical theory of the next subsection; this shows how tentative and preliminary status the present subsection has.

as φ_1 is. Then φ_2 also sends γ into γ' . Since G_{as} can be identified with the factor group G/G_0 , this shows that the asymptotic symmetry group acts on the physical phase space.

Consider a surface σ in $\tilde{\Gamma}$ that cuts each orbit exactly once. Such a surface is called a *section* of the quotient $\tilde{\Gamma}/G_0$. Each section of $\tilde{\Gamma}/G_0$ breaks the gauge group and can, therefore, be called *covariant gauge fixing*. A section σ can also be described as follows. Let X^μ be coordinates on \mathcal{M} and o^A coordinates on Γ . Then, for the special case that $\Psi = g$, σ determines the set of functions

$$g_{\mu\nu}(o, X), \quad (1)$$

that is, metric components with respect to the coordinates X^μ at the point g of intersection between σ and the orbit given by the coordinates o^i . The coordinates X^μ can be arbitrary and the same section can so be represented by different sets of functions (1). That is why the gauge fixing is called “covariant.” Observe also that no global coordinate chart is necessary to describe σ .

Transformations between two different gauge fixings is constituted by a set of diffeomorphisms $\varphi(o)$, one for each orbit γ determined by coordinates o^i . Such gauge transformations are common in every gauge theory. For example, in electrodynamics, one often uses the so-called Coulomb gauge. This is defined by a differential equation for the components of the potential. One also uses the so-called axial gauge that is defined by an algebraic equation for the potential. The transformation between these two gauges must, therefore, depend on the potential in a rather complicated and non-local way. Quantum field theory constructed in a given gauge *cannot* be made invariant with respect to all field dependent gauge transformations. One hopes to overcome this difficulty by working with gauge-invariant quantities.

Similarly, for generally covariant systems, the two quantum theories constructed in two different gauges that are related by a field dependent transformation are not unitarily equivalent, unless they are limited to relations between Dirac observables [43].

Let $o \in \Gamma$ and $\Psi(o) = \sigma \cap \pi_{\text{sol}}^{-1} o$. It is a simple exercise to show that there is an element φ_o in each class of G/G_0 such that $\varphi_{o*} \Psi(o) \in \sigma$; if $\Psi(o)$ does not admit any spacetime symmetry, then φ_o is even unique! Thus, given a gauge fixing, we find a (o -dependent) extension of each asymptotic symmetry to the whole of \mathcal{M} . Even if gauge (and, in general, o -) dependent, this extension is a practical tool that will be used later.

2.2 Phase space

We assume now that \mathcal{M} has the structure $\Sigma \times \mathbb{R}$ and that the dynamical equation admits a well-posed Cauchy problem on the manifold Σ . Let us denote a Cauchy

datum for the solution Ψ by $(\Psi_\Sigma, \Pi_\Sigma)$, where Ψ_Σ and Π_Σ are some geometric objects on Σ . We assume that a Riemannian metric ${}^3g_{kl}$ with the meaning of the first fundamental form of Σ belongs to Ψ_Σ and a symmetric tensor field K_{kl} with the meaning of the second fundamental form of Σ belongs to Π_Σ . We assume further that $(\Sigma, {}^3g)$ is asymptotically Euclidean. This involves the existence of a special coordinate patch in Σ such that the corresponding components of all objects within Ψ_Σ and Π_Σ satisfy suitable fall-off asymptotic conditions, so-called *Cauchy data asymptotic conditions* (CDAC). Different examples of CDAC are given in [9] and in [8], Sec. 5.4.

For generally covariant systems, the objects Ψ_Σ and Π_Σ must satisfy some particular conditions (mostly differential equations) in order to constitute an initial datum for a solutions. These equations are called *constraints*.

The *phase space* \mathcal{P} of the system is a manifold the points of which are all unconstrained Cauchy data. It carries a symplectic structure $\bar{\Omega}$, which defines Poisson brackets. We assume that such a space can be constructed. For example, ${}^3g_{kl}$ and

$$\pi^{kl} := \sqrt{\text{Det}({}^3g)}({}^3g^{kl} {}^3g^{rs} - {}^3g^{kr} {}^3g^{ls})K_{rs}$$

are canonically conjugated variables for the case that $\Psi = g$ [10].

We assume further that all constrained data form a submanifold \mathcal{C} of \mathcal{P} , called *constraint surface*: each point of \mathcal{C} is a Cauchy datum for a solution and each Cauchy datum for a solution lies in \mathcal{C} .

There is an important relation “ \sim ” between points of \mathcal{C} that can be defined as follows. The points p_1 and p_2 are said to satisfy $p_1 \sim p_2$ if the solutions Ψ_1 and Ψ_2 determined by them satisfy

$$\Psi_2 = \varphi_* \Psi_1$$

for some $\varphi \in G_0$. We assume that “ \sim ” is an equivalence relation. (For general relativity, this has been shown in [11]). The equivalence class of points at \mathcal{C} are called *c-orbits* and denoted by λ . Each c-orbit determines a class of G_0 -equivalent solutions and vice versa. Hence, the quotient \mathcal{C}/λ is the physical phase space defined in Sec. 2.1:

$$\Gamma = \mathcal{C}/\lambda.$$

The quotient projection of the constraint surface \mathcal{C} to the physical phase space Γ will be denoted by π_{phs} .

Let the constraint functionals

$$\mathcal{H}[\Psi_\Sigma, \Pi_\Sigma; x), \mathcal{H}_k[\Psi_\Sigma, \Pi_\Sigma; x),$$

where $k = 1, 2, 3$ and $x \in \Sigma$, define the constraint surface \mathcal{C} by $\mathcal{H} = 0$ and $\mathcal{H}_k = 0$. It seems that they can be chosen for all generally covariant systems so that they obey the so-called Dirac algebra [12], see also [13]: Let x^k be coordinates, $\mathcal{N}(x)$ a scalar and $\mathcal{N}^k(x)$ a vector fields on Σ ; the fields $\mathcal{N}(x)$ and $\mathcal{N}^k(x)$ are called *lapse* and *shift*, respectively. Let $\vec{\mathcal{N}} := (\mathcal{N}, \mathcal{N}^1, \mathcal{N}^2, \mathcal{N}^3)$ and

$$\mathcal{H}[\vec{\mathcal{N}}] := \int_{\Sigma} d^3x (\mathcal{N}(x)\mathcal{H}(x) + \mathcal{N}^k(x)\mathcal{H}_k(x)) .$$

Then

$$\left\{ \mathcal{H}[\vec{\mathcal{N}}_1], \mathcal{H}[\vec{\mathcal{N}}_2] \right\} = \mathcal{H}[\vec{\mathcal{N}}] , \quad (2)$$

where

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_1^k \partial_k \mathcal{N}_2 - \mathcal{N}_2^k \partial_k \mathcal{N}_1 , \\ \mathcal{N}^k &= \mathcal{N}_1^l \partial_l \mathcal{N}_2^k - \mathcal{N}_2^l \partial_l \mathcal{N}_1^k + {}^3g^{kl} (\mathcal{N}_1 \partial_l \mathcal{N}_2 - \mathcal{N}_2 \partial_l \mathcal{N}_1) . \end{aligned}$$

This implies that the Poisson brackets of the constraints vanish at the constraint surface; such a constraint surface is called *first class* [14], [15]. However, Eq. (2) holds only if the lapse and shift fields satisfy some fall-off conditions (see [9]), called *gauge group asymptotic conditions* (GGAC). An unexpected result of [9] is that $\vec{\mathcal{N}}$ need not approach zero asymptotically in order to satisfy (GGAC) but can contain arbitrary odd-parity supertranslations.

The lapse and shift fields can even approach linear functions of coordinates asymptotically. This corresponds to infinitesimal Poincaré transformations at infinity (cf. [16] and [9]). The condition that $\vec{\mathcal{N}} \rightarrow \vec{\mathcal{N}}_{\infty}$, where $\vec{\mathcal{N}}_{\infty}$ represents the asymptotic behaviour of $\vec{\mathcal{N}}$ corresponding to an element of the Lie algebra of Poincaré group is called *Poincaré group asymptotic condition* (PGAC); the form of $\vec{\mathcal{N}}_{\infty}$ as a function on Poincaré algebra is given in [16] and [9]. The corresponding functionals $\mathcal{H}[\vec{\mathcal{N}}]$ are not differentiable (their variations lead to surface integrals at infinity) or are not even convergent. This can be improved by adding *surface terms* at infinity to them that we denote by $\mathcal{H}_{\infty}[\vec{\mathcal{N}}_{\infty}]$: the functional form of $\mathcal{H}_{\infty}[\vec{\mathcal{N}}_{\infty}]$ is given in [16] and [9]. An analysis of similar surface terms for the null-infinity is given in [8]. The expressions $\mathcal{H}[\vec{\mathcal{N}}] + \mathcal{H}_{\infty}[\vec{\mathcal{N}}_{\infty}]$ have finite values on constraint surface and generate asymptotic Poincaré transformation³.

Observe that the CDAC form a part of the definition of the phase space, while GGAC and PGAC help define some transformations on this phase space. In particular, the CDAC must be preserved in all transformations satisfying the GGAC or

³The surface terms may be uniquely determined (except for additive constants) by boundary conditions imposed on the dynamics of the fields at the surface. For a finite surface, this has been shown by [17].

PGAC, see [9]. In this way, the set of all lapse and shift fields is divided into those that are associated with infinitesimal gauge transformations, infinitesimal dynamical symmetries and the rest that is not interesting. Again, the gauge \vec{N} 's form a large subset of the symmetry \vec{N} 's. What is the relation between this Cauchy surface picture and the spacetime picture of subsection 2.1? We believe that such a relation can be found using the technique invented by Kuchař, the so-called Kuchař variables (embeddings, see subsection 2.4). In particular, the CDAC must be related to asymptotic conditions imposed on the embeddings. An example of how this may work will be given in the next section.

If \vec{N} satisfies the GGAC, the transformation generated by $\mathcal{H}[\vec{N}]$ in the phase space shifts the points of \mathcal{C} along c-orbits, and any point of a given c-orbit can be reached in this way from any other point of it. This is the reason to call λ a c-orbit (constraints orbit).

For any first class constraint surface, there is a unique way of how the symplectic structure $\bar{\Omega}$ of the phase space \mathcal{P} defines a symplectic structure Ω on the physical phase space Γ : First step is to pull back the two-form $\bar{\Omega}$ to \mathcal{C} . The result is a two-form $\tilde{\Omega}$. The form $\tilde{\Omega}$ is degenerated along c-orbits; it is not a symplectic form, but it is closed ($d\tilde{\Omega} = 0$, where d denotes the external differentiation). The second step is a proof that there is a unique symplectic form Ω on Γ such that the pull back of it by the projection π_{phs} to \mathcal{C} is $\tilde{\Omega}$. In this way, we finally obtain the full-fledged physical phase space (Γ, Ω) .

2.3 Observables and dynamical symmetries

The action of the gauge group has been quotiented away from the physical phase space, so it is a gauge-invariant structure. Functions on it are called *observables*:

$$o : \Gamma \mapsto \mathbb{R} .$$

The central idea of the gauge-invariant method described by these lectures is to answer all physically interesting questions by manipulating quantities defined on the physical phase space, such as observables. To interpret or justify such manipulations, we need the connection of Γ to the larger spaces \mathcal{C} , \mathcal{P} and $\tilde{\Gamma}$.

Each observable o determines a function \tilde{o} on \mathcal{C} by

$$\tilde{o} := o \circ \pi_{\text{phs}} . \tag{3}$$

Clearly, all such functions must be constant along c-orbits:

$$\left\{ \tilde{o}, \mathcal{H}[\vec{N}] \right\} = 0$$

for all \vec{N} that satisfy GGAC; they are called *Dirac observables* [14]. Examples of Dirac observable are the quantities $\mathcal{H}[\vec{N}] + \mathcal{H}_\infty[\vec{N}_\infty]$ for all fields \vec{N} that satisfy PGAC, see [9].

Let $\{o^i\}$ be some coordinates on Γ ; they form a *complete set of observables*. Similarly, the corresponding functions \tilde{o}^i form *complete set of Dirac observables*. One needs complete sets with some additional properties to quantize by a gauge-invariant way, and one needs still more observables to describe all interesting properties of the quantum system. This will be shown later by an example. The idea of basing quantization of gravity on complete sets of Dirac observables encounters a difficulty: there is no single quantity of this kind known in general relativity, for example in the case when the Cauchy surface is compact [18], [19]. Even if it were known, it would be non-local [20]. In the asymptotically flat case, however, two complete sets are known: they are associated with the asymptotic in- and out-fields. They have been described, within a perturbation theory, by DeWitt (“asymptotic invariants”) [21], and for the exact theory by Ashtekar (“radiative modes”) [22]. We propose to work out a theory for this favorable case first and then to see if anything can be done for the other cases.

Let $\bar{h} : \mathcal{P} \mapsto \mathcal{P}$ be a map that preserves the symplectic form $\bar{\Omega}$ (symplectomorphism) and the constraint surface \mathcal{C} in \mathcal{P} . Such a map is called *extended dynamical symmetry* (extended to the extended phase space \mathcal{P}). We use the word “dynamical” to distinguish it from spacetime symmetries.

Clearly, such a map \bar{h} induces a map of the constraint surface onto itself: $\tilde{h} := \bar{h}|_{\mathcal{C}}$ and $\tilde{h} : \mathcal{C} \mapsto \mathcal{C}$. The most important properties of \tilde{h} are: it preserves the form $\tilde{\Omega}$ and maps the c-orbits onto c-orbits. Such a map \tilde{h} is called *dynamical symmetry*. Each dynamical symmetry \tilde{h} induces a map $h : \Gamma \mapsto \Gamma$ that preserves the symplectic structure Ω , so h is a symplectomorphism of the physical phase space (Γ, Ω) .

An infinitesimal extended dynamical symmetry $\overline{d\bar{h}}$ is generated via Poisson brackets by a function $-\bar{H}dt : \mathcal{P} \mapsto \mathcal{P}$ (the sign is chosen for later convenience). The restriction \tilde{H} of \bar{H} to \mathcal{C} is a Dirac observable and so it defines a unique function H on the physical phase space Γ . The function $-Hdt$ generates the infinitesimal symplectomorphism dh in Γ that is induced by $\overline{d\bar{h}}$, for proofs, see [9], [3] and [4]. (We denote infinitesimal maps by a symbol composed from “ d ” and a letter; the letter itself has no further meaning.)

The infinitesimal transformations generated by $(\mathcal{H}[\vec{N}] + \mathcal{H}_\infty[\vec{N}_\infty])dt$ are dynamical symmetries for all \vec{N} satisfying PGAC. Their Poisson brackets at the constraint surface form the Lie algebra of Poincaré group [9]. They have in general non-vanishing Poisson brackets with other Dirac observables generating a non-trivial dynamics for them. We shall give an interpretation of this dynamics in the next two subsections.

2.4 Transversal surfaces

A transversal surface \mathcal{T} is a submanifold of \mathcal{C} that intersects each c-orbit exactly once and transversally; \mathcal{T} is a section of the quotient \mathcal{C}/λ . Each transversal surface inherits a two-form $\Omega_{\mathcal{T}}$ from \mathcal{C} by pull back of $\tilde{\Omega}$ to \mathcal{T} by the injection map of \mathcal{T} into \mathcal{C} . It is not only closed but also non-degenerate, so $(\mathcal{T}, \tilde{\Omega})$ is a symplectic manifold (see also [9], where transversal surface is called “gauge condition”).

The projection π_{phs} maps \mathcal{T} to Γ and its restriction $\pi_{\text{phs}}|_{\mathcal{T}}$ is actually a bijection having an inverse. The map $\pi_{\text{phs}}|_{\mathcal{T}}$ can be shown to be a symplectomorphism of the spaces $(\mathcal{T}, \Omega_{\mathcal{T}})$ and (Γ, Ω) . Thus, each transversal surface is a “model” of the physical phase space. If we wish to calculate Ω in the coordinates $\{o^i\}$, then we just have to calculate $\Omega_{\mathcal{T}}$ in the coordinates $\{\tilde{o}^i\}$.

An important issue is the relation between a transversal surface \mathcal{T} and Cauchy surfaces in solution space-times. Such a relation is provided by a gauge fixing σ . As it has been shown above, σ can be represented as a set of functions $\Psi(o, X)$, where $\{o^i\}$ are coordinates on Γ and X^μ coordinates on \mathcal{M} . Consider the space-time $(\mathcal{M}, \Psi(o, X))$; a Cauchy surface in such a spacetime can be described by an embedding $\hat{X}(o) : \Sigma \mapsto \mathcal{M}$ given by the set of functions $X^\mu(o, x)$. Having these functions and the solution $\Psi(o, X)$, we can calculate the corresponding Cauchy datum $(\Psi_{\hat{X}(o)}, \Pi_{\hat{X}(o)})$ and so determine a point at \mathcal{C} . More specifically, this point lies at the c-orbit $\lambda(o)$ corresponding to the point $o \in \Gamma$: $\lambda(o) = \pi_{\text{phs}}^{-1}o$. In this way, there is a unique point $(\Psi_{\hat{X}(o)}, \Pi_{\hat{X}(o)})$ at \mathcal{C} for each λ , and this defines a transversal surface \mathcal{T} .

Let us denote the resulting map that sends the set of embeddings $\{\hat{X}(o)\}$ to the transversal surface \mathcal{T} by χ_σ :

$$\chi_\sigma\{\hat{X}(o)\} = \mathcal{T} .$$

The functions $\Psi(o, X)$ and $X(o, x)$ must be sufficiently smooth or else \mathcal{T} would be rather jumpy.

One can show that χ_σ can even be a bijection between the space of embedding sets $\{X(o, x)\}$ and the space of transversal surfaces \mathcal{T} . The condition is that the solutions $\Psi(o, X)$ do not admit any spacetime symmetry [5]. Then each transversal surface and σ determine together a set of embeddings $\{X(o, x)\}$, one embedding for each solution spacetime $(\mathcal{M}, \Psi(o, X))$:

$$\{X(o, x)\} = \chi_\sigma^{-1}\mathcal{T} .$$

One can say that, given a gauge, each transversal surface defines a *many-finger-time* instant in each solution spacetime. Transversal surfaces can, therefore, also be called many-finger-time levels.

An interesting point is that the variables o^i and $\{X(o, x)\}$ —coordinates on the physical phase space and a set of embeddings, one for each solution—form a coordinate system on \mathcal{C} if a gauge σ is specified. They are called generalized⁴ *Kuchař variables*, and they constitute a useful tool for many calculations because they provide a neat division of variables into gauge, physical and dependent degrees of freedom on one hand and a bridge between the four-dimensional and the three-dimensional pictures on the other.

2.5 Time evolution

Dirac observables are constant along whole solutions and so they are not only gauge invariant but a kind of integrals of motion. It seems that their dynamics is trivial: they just stay constant. Bergmann [18] characterized that as “frozen dynamics”.

However, as early as 1949, Dirac [25] has put forwards a theory of time evolution that is based on transversal surfaces and symmetries and that leads to a non trivial time evolution of Dirac observables. The idea is that a physically sensible evolution results if the “pure” dynamics (such as staying constant) is compared with a fiducial “zero” dynamics defined by a symmetry. The account presented here is a generalization [3], [4] of Dirac theory. It can also be understood as an illustration of the discovery [26] that the Hamiltonian dynamics needs a frame of reference and that different frames lead to differently looking time evolutions of one and the same system, see also [8].

Let \tilde{h} be dynamical symmetry and \mathcal{T} a transversal surface. Then $\tilde{h}\mathcal{T} = \mathcal{T}'$ is another transversal surface and \tilde{h} is a symplectomorphism between the spaces $(\mathcal{T}, \Omega_{\mathcal{T}})$ and $(\mathcal{T}', \Omega_{\mathcal{T}'})$. Thus, \tilde{h} can shift many-finger-time levels.

Suppose a one-dimensional group of extended dynamical symmetries $\bar{h}(t)$ is given; t is the parameter of the group, $\bar{h}(t_1) \cdot \bar{h}(t_2) = \bar{h}(t_1 + t_2)$. Let the generator of the group be $-\bar{H}$, and let the induced groups and generators on \mathcal{C} and Γ be denoted by $\tilde{h}(t)$, $h(t)$ and $-H$. We use the group $\tilde{h}(t)$ to build up a reference system in \mathcal{C} with respect to which we shall describe the motion represented by c-orbits in a similar way as the particle world lines in Minkowski spacetime are described by their coordinates in an inertial frame.

Let \mathcal{T} be a transversal surface. Define $\mathcal{T}_t := \tilde{h}(t)\mathcal{T}_0$. These surfaces form a one-dimensional family that we denote by $\{\mathcal{T}_t\}$. The family determines a subset $\eta_{\lambda}(t)$ of any c-orbit λ by

$$\eta_{\lambda}(t) := \mathcal{T}_t \cap \lambda .$$

The curve $\eta_{\lambda}(t)$ lies in λ and is one-dimensional in spite of λ being itself many (or

⁴The embeddings introduced originally by Kuchař [23], [24] were o -independent.

even infinitely many) dimensional. We call $\eta_\lambda(t)$ a trajectory of the system with respect to the family $\{\mathcal{T}_t\}$.

Let \tilde{o}_0 be a Dirac observable representing some measurement done at the time level⁵ \mathcal{T}_0 . The values of the function $\tilde{o}_0|_{\mathcal{T}_0}$ on \mathcal{T}_0 give results of measurement at each point of \mathcal{T}_0 , that is, at each instantaneous state of the system.

We define *the same measurement* at the time \mathcal{T}_t to be represented by Dirac observable $\tilde{o}_t := \tilde{o}_0 \circ \tilde{h}(-t)$. The function \tilde{o}_t is the image of \tilde{o}_0 by the map $\tilde{h}(t)$: it gives the same results at the states that are related by the symmetry:

$$\tilde{o}_t(\tilde{h}(t)p) = \tilde{o}_0(p)$$

for any point $p \in \mathcal{C}$. Actually, the set $\{\tilde{o}_t|t \in \mathbb{R}\}$ of Dirac observables is a special case of Rovelli's "evolving constants of motion" [27].

Now, we are ready to define the time evolution.

Definition 1 *Time evolution is the change in the results of the same measurement done at different times along a dynamical trajectory of the system.*

Clearly, these results are given by the function $\tilde{o}_t(\eta_\lambda(t))$ for the measurement represented by $\{\tilde{o}_t|t \in \mathbb{R}\}$ and the trajectory λ . The function can be written in two ways.

Schrödinger picture By substituting from the definition of \tilde{o}_t , we have

$$\tilde{o}_t(\eta_\lambda(t)) = \tilde{o}_0(\tilde{h}(-t)\eta_\lambda(t)) .$$

Let us define

$$\tilde{\xi}_\lambda(t) := \tilde{h}(-t)\eta_\lambda(t) ;$$

$\tilde{\xi}_\lambda(t)$ is a curve in \mathcal{T}_0 and can be viewed as a "time-dependent state of the system." Then we can write

$$\tilde{o}_t(\eta_\lambda(t)) = \tilde{o}_0|_{\mathcal{T}_0}(\tilde{\xi}_\lambda(t)) . \tag{4}$$

The measurement is represented by a single (time-independent) Schrödinger observable $\tilde{o}_0|_{\mathcal{T}_0}$.

⁵Indeed, a given Dirac observable cannot, in general, be measured anywhere on the constraint surface but only in a neighbourhood of a transversal surface that belongs to the definition of the observable, see [4].

Heisenberg picture Since \tilde{o}_t is a Dirac observable for each t , it is constant along c-orbits, and we find

$$\tilde{o}_t(\eta_\lambda(t)) = \tilde{o}_t(\eta_\lambda(0)) .$$

All quantities can again be taken at \mathcal{T}_0 :

$$\tilde{o}_t(\eta_\lambda(t)) = \tilde{o}_t|_{\mathcal{T}_0}(\eta_\lambda(0)) . \quad (5)$$

Now, the state is described by a single point $\eta_\lambda(0)$ at \mathcal{T}_0 (time independence) while the measurement is described by a set of functions $\tilde{o}_t|_{\mathcal{T}_0}$ on \mathcal{T}_0 , constituting a time-dependent Heisenberg observable $\{\tilde{o}_t|_{\mathcal{T}_0} | t \in \mathbb{R}\}$.

The dynamical equation for the Schrödinger state $\tilde{\xi}(t)$ or the Heisenberg observable $\{\tilde{o}_t|_{\mathcal{T}_0} | t \in \mathbb{R}\}$ can be calculated with the following results: The Schrödinger state $\tilde{\xi}(t)$ is an integral curve in $(\mathcal{T}_0, \Omega_{\mathcal{T}_0})$ of the Hamiltonian vector field $d\tilde{H}|_{\mathcal{T}_0}^\#$ of $\tilde{H}|_{\mathcal{T}_0}$:

$$\frac{d\tilde{\xi}_\lambda(t)}{dt} = d\tilde{H}|_{\mathcal{T}_0}^\# . \quad (6)$$

The Heisenberg observable $\{\tilde{o}_t|_{\mathcal{T}_0}\}$ must satisfy the differential equation

$$\frac{d\tilde{o}_t|_{\mathcal{T}_0}}{dt} = \{d\tilde{o}_t|_{\mathcal{T}_0}, \tilde{H}|_{\mathcal{T}_0}\}_{\Omega|_{\mathcal{T}_0}} , \quad (7)$$

where $\{\cdot, \cdot\}_{\Omega|_{\mathcal{T}_0}}$ is the Poisson bracket of $(\mathcal{T}_0, \Omega_{\mathcal{T}_0})$.

All definitions and equations hitherto written down depend on the choice of \mathcal{T}_0 . This leads to the impression that our dynamics depends not only on the chosen symmetry group $\tilde{h}(t)$ but also on \mathcal{T}_0 . This impression may be strengthened if one reads Dirac's paper. However, Dirac's choice of the group was related to his choice of the surface. Here, these two choices are independent.

Actually, the exposition given as yet has aimed at some physical interpretation of the final result, which comes only now: The symplectomorphism $\pi_{\text{phs}}|_{\mathcal{T}_0}$ maps the Eqs. (6) and (7) at \mathcal{T}_0 to the following equations at Γ

$$\frac{d\xi_\lambda}{dt} = dH^\# \quad (8)$$

and

$$\frac{do_t}{dt} = \{o_t, H\}_\Omega , \quad (9)$$

where

$$\xi_\lambda(t) := \pi_{\text{phs}}|_{\mathcal{T}_0} \tilde{\xi}_\lambda(t) , \quad \tilde{o}_t|_{\mathcal{T}_0} = o_t \circ \pi_{\text{phs}}|_{\mathcal{T}_0} .$$

Eqs. (8) and (9) have the same form for any choice of \mathcal{T}_0 and depend only on the group $h(t)$.

The symmetry group $h(t)$ is to be chosen as a suitable subgroup of the asymptotic symmetries. To be sure, asymptotic symmetries cannot be extended to the whole of \mathcal{M} in a unique way (a gauge-dependent extension has been constructed at the end of subsection 2.1). However, the action of such extended map on a gauge-invariant quantity does not depend on the way it has been extended. The dynamics constructed in this way is, therefore, gauge invariant and it seems to be physically sound.

3 A model: gravitating shell

This section will illustrate the notions and methods introduced in the preceding one. At the same time, the dynamics of the shell will be given a form that will be suitable for quantization.

3.1 Space of solutions, gauge group and asymptotic symmetries

The model consists of a spherically symmetric thin shell of light-like matter surrounded by its own gravitational field. Hence, there are two geometrical objects in $\bar{\Psi}$: a metric $g_{\mu\nu}$ describing the gravitational field and a three-dimensional light-like surface \bar{S} that is a trajectory of the shell. The background manifold $\bar{\mathcal{M}}$ is $\mathbb{R}^3 \times \mathbb{R}$, where \mathbb{R}^3 is the manifold of Cauchy surface. The dynamical equations are Einstein's and matter equations. Any solution $\bar{\Psi}$ can be constructed by sticking together a piece of Schwarzschild solution of mass parameter M with the meaning of gravitational radius, and a piece of Minkowski spacetime, along a spherically symmetric null hypersurface so that the points with the same radius coordinate cover each other (see, e.g., [6]).

The spherical symmetry enables us to reduce the dimension of the problem by two. The effective spacetime can be considered as the space of the rotation group orbits; the background manifold \mathcal{M} is $\mathbb{R} \times \mathbb{R}_+$, where $\mathbb{R}_+ := (0, \infty)$. The four-dimensional metric decomposes into a two-dimensional metric g_{AB} on \mathcal{M} and a scalar field R on \mathcal{M} (radius of the rotation group orbit). The shell trajectory becomes a curve S . These are the three geometrical objects in Ψ .

Luckily, for this simple model, all solutions can be explicitly written down. Thus, we can make a list of all physically different solutions to a basis of our analysis. The list can best be given in a particular gauge determined by a covariant gauge fixing σ . To specify σ , we choose the coordinates U and V on \mathcal{M} with ranges $U \in \mathbb{R}$

and $V \in (U, \infty)$. The boundary of \mathcal{M} (that does not belong to it) consists of three pieces: $\partial_0\mathcal{M}$ given by the equation $V = U$, with a coordinate $T_0 := (U + V)/2$, $T_0 \in \mathbb{R}$, $\partial_+\mathcal{M}$ defined by the limit $V \rightarrow \infty$ and described by coordinate $U_\infty = U$, $U_\infty \in \mathbb{R}$, and $\partial_-\mathcal{M}$ defined by the limit $U \rightarrow -\infty$ and described by coordinate $V_\infty = V$, $V_\infty \in \mathbb{R}$. With respect to the fixed coordinates U and V , we set conditions on the components of the representative metric, scalar field and shell trajectory as follows (for more detail and motivation, see [6] and [28]).

1. The coordinates U and V are double null coordinates:

$$ds^2 = -A(U, V)dUdV .$$

2. For out-going shells, S is defined by $U = \text{const}$, and the coordinate U coincides with a retarded time at $\partial_+\mathcal{M}$.
3. For in-going shells, S is defined by $V = \text{const}$, and the coordinate V coincides with an advanced time at $\partial_-\mathcal{M}$.
4. The functions $A(U, V)$ and $R(U, V)$ are continuous at the shell.

Then the representative solutions can be described as follows (see [6] and [28]). We introduce a parameter η with two values $+1$ and -1 to define the direction of radial motion of the shell: expanding for $\eta = +1$ and contracting for $\eta = -1$.

Out-going shell ($\eta = +1$).

1. The shell trajectory is given by

$$U = w ,$$

$w \in \mathbb{R}$ being a measurable parameter: the retarded arrival time of the out-going shell.

2. Left from the shell, $U > w$, the two-metric $A(U, V)$ and the scalar field $R(U, V)$ are given by

$$A = 1 , \quad R = \frac{-U + V}{2} . \tag{10}$$

3. Right from the shell, $U < w$, they are given by

$$A = \frac{1}{\kappa(f_+)e^{\kappa(f_+)}} \frac{-w + V}{4M} \exp\left(\frac{-U + V}{4M}\right) , \tag{11}$$

and

$$R = 2M\kappa(f_+) , \quad (12)$$

where

$$f_+ = \left(\frac{-w + V}{4M} - 1 \right) \exp \left(\frac{-U + V}{4M} \right) . \quad (13)$$

The Kruskal function κ is defined by its inverse

$$\kappa^{-1}(x) = (x - 1)e^x$$

on the interval $x \in (0, \infty)$.

At the shell,

$$\lim_{U \rightarrow w_-} f_+ = \kappa^{-1} \left(\frac{-w + V}{4M} \right) ,$$

hence, we have $R \rightarrow (-w + V)/2$ and $A \rightarrow 1$ and the metric is continuous, but its derivatives have a jump. In this way, these coordinates carry the differential structure C^1 that is determined by the metric. The stress-energy tensor of the shell matter can be calculated by the formula given in [29]. Let us denote this solution by $\Psi_+(M, w; U, V)$.

The boundary $R \rightarrow 0$ is given by $V \rightarrow U$ for $U > w$ and by $f_+ \rightarrow -1$ or

$$V \rightarrow w + 4M\kappa \left[-\exp \left(\frac{U - w}{4M} \right) \right] \quad (14)$$

for $U < w$. The curve defined by the right-hand side of (14) is space-like running from the point $(U, V) = (w, w)$ to the point $(U, V) = (-\infty, w + 4M)$. It is the boundary of the open domain \mathcal{D}_Ψ in \mathcal{M} where the representative solution (\mathcal{M}, Ψ_+) is well defined. The curve $V = w + 4M$ is the horizon, $R = 2M$.

In-going shell ($\eta = -1$).

1. The shell trajectory is given by

$$V = w ;$$

$w \in \mathbb{R}$ is the advanced departure time of the in-going shell.

2. Left from the shell, $V < w$, the two-metric $A(U, V)$ and the scalar field $R(U, V)$ are given by

$$A = 1 , \quad R = \frac{-U + V}{2} . \quad (15)$$

3. Right from the shell, $V > w$, they are given by

$$A = \frac{1}{\kappa(f_-)e^{\kappa(f_-)}} \frac{-U + w}{4M} \exp\left(\frac{-U + V}{4M}\right) , \quad (16)$$

and

$$R = 2M\kappa(f_-) , \quad (17)$$

where

$$f_- = \left(\frac{-U + w}{4M} - 1\right) \exp\left(\frac{-U + V}{4M}\right) . \quad (18)$$

Let us denote this solution by $\Psi_-(M, w; U, V)$. Again, the metric is continuous at the shell. The boundary of the domain \mathcal{D}_Ψ is given by the equation $f_- = -1$. Observe that the solutions $\Psi_-(M, w; U, V)$ fill in the lower corner of \mathcal{M} and so every point of the manifold \mathcal{M} is used by both solutions $\Psi_+(M, w; U, V)$ and $\Psi_-(M, w; U, V)$.

The first employment of our list is the specification of the physical phase space. It has two components, Γ_+ and Γ_- corresponding to $\eta = +1$ and $\eta = -1$. They have both the manifold structure of $\mathbb{R}_+ \times \mathbb{R}$, where $M \in \mathbb{R}_+$ and $w \in \mathbb{R}$. M and w can serve as coordinates on Γ_η . It is not yet clear that these coordinates are regular; this ought to follow from the regularity of the symplectic form of the physical phase space written with respect of these coordinates. The symplectic form cannot be inferred from our list of solutions and will be calculated later.

The group of general four-dimensional diffeomorphisms can also be reduced by the symmetry: we admit only diffeomorphisms that commute with rotations so that the rotation group orbits are mapped onto such orbits. Clearly, regular centre points are sent only to regular centre points—so that the regular centre is invariant—while a spherical orbit can be sent to any spherical orbit. Hence, the reduced group is $\text{Diff}\mathcal{M}$ for our two-dimensional background manifold \mathcal{M} .

The second employment of the solution list given above is to find symmetries of the system. The symmetries will help us to decompose the group $\text{Diff}\mathcal{M}$, to define the most important observables and to construct the dynamics. To find the symmetries, we observe that some pairs of our representative solutions are isometric or conformally related to each other. For each such pair, there is a unique element of $\text{Diff}\mathcal{M}$ that implements the relation.

Time shifts Clearly, the solutions $\Psi_\eta(M, w; U, V)$ and $\Psi_\eta(M, w + t; U, V)$ are isometric for any $t \in \mathbb{R}$, M , w and η . The corresponding diffeomorphism $\varphi_H(t)$ is given by

$$\varphi_H(t) : (U, V) \mapsto (U + t, V + t) \quad (19)$$

For example, if $\eta = +1$, the shell is sent to $U = w + t$, the new metric and the scalar are, for $U > w + t$,

$$A = 1, \quad R = \frac{-U + V}{2},$$

while, for $U < w + t$,

$$A = \frac{1}{\kappa(f'_+)e^{\kappa(f'_+)}} \frac{-w - t + V}{4M} \exp\left(\frac{-U + V}{4M}\right),$$

and

$$R = 2M\kappa(f'_+),$$

where

$$f'_+ = \left(\frac{-w - t + V}{4M} - 1\right) \exp\left(\frac{-U + V}{4M}\right).$$

Hence, these are just the fields (10)–(13) with $w \mapsto w + t$. The proof for $\eta = -1$ is analogous. Observe that the map $\varphi_H(t)$ is well defined everywhere on \mathcal{M} and independent of the parameters η , M and w . Hence, all such maps form a one-dimensional group G_H with parameter t . The group acts on the physical phase space: $\varphi_H(t) : (M, w) \mapsto (M, w + t)$. Finally, its action can be uniquely extended to the boundary:

$$U_\infty \mapsto U_\infty + t, \quad V_\infty \mapsto V_\infty + t, \quad T_0 \mapsto T_0 + t.$$

Dilatations Clearly, the solutions $\Psi_\eta(M, w; U, V)$ and $\Psi_\eta(e^s M, e^s w; U, V)$ are conformally related for all $s \in \mathbb{R}$, η , M and w . On the other hand, they are sent into each other by a composite map $(\varphi_D(s), \omega(\varphi_D(s)))$, where

$$\varphi_D(s) : (U, V) \mapsto (e^s U, e^s V) \tag{20}$$

is a diffeomorphism and

$$\omega(\varphi_D(s)) : g_{\mu\nu} \mapsto e^{2s} g_{\mu\nu}$$

is a constant conformal deformation (CCD) that acts only on the (four-dimensional) metric: it scales the two-dimensional metric g_{AB} by e^{2s} and the scalar field R by e^s .

Actually, the dilatation is nothing but a rescaling: all quantities with a dimension are rescaled by the corresponding power of e^s . The coordinates U and V have acquired their scale from the metric through the gauge fixing. The Newton constant G must also be rescaled: $G \mapsto e^{2s} G$!

To see that the map (20) makes the job, let us restrict ourselves to the case $\eta = +1$ and study the action of the map on the objects in the spacetime defined by Eqs. (10)–(13). The metric is transformed by $\varphi_D(s)_*$ to

$$ds^2 = -e^{-2s} A(e^{-s}U, e^{-s}V) dU dV ,$$

the scalar field to

$$R = R(e^{-s}U, e^{-s}V)$$

and the shell position changes as follows

$$w \mapsto e^s w .$$

Thus, for $U > e^s w$, the new metric A is e^{-2s} and the new scalar R is

$$e^{-s} \frac{-U + V}{2} ;$$

for $U < e^s w$, the new metric A is

$$A = \frac{e^{-2s}}{\kappa(f'_+) e^{\kappa(f'_+)}} \frac{-we^s + V}{4Me^s} \exp\left(\frac{-U + V}{4Me^s}\right)$$

and the scalar R ,

$$R = 2M\kappa(f'_+) ,$$

where

$$f'_+ = \left(\frac{-we^s + V}{4Me^s} - 1 \right) \exp\left(\frac{-U + V}{4Me^s}\right) .$$

The subsequent CCD by e^{2s} brings A and R to

$$A = 1 , \quad R = \frac{-U + V}{2} \tag{21}$$

for $U > we^s$, to

$$A = \frac{1}{\kappa(f'_+) e^{\kappa(f'_+)}} \frac{-we^s + V}{4Me^s} \exp\left(\frac{-U + V}{4Me^s}\right) \tag{22}$$

and

$$R = 2Me^s \kappa(f'_+) , \tag{23}$$

for $U < we^s$, and the shell stays at $U = we^s$. But the fields (21)–(23) are just the same as (10)–(13) except for M being changed to Me^s and w to we^s .

We observe that the map $\left(\varphi_D(s), \omega(\varphi_D(s))\right)$ is well defined on the whole \mathcal{M} and independent of η , M and w . Since each CCD commutes with any diffeomorphism, the dilatations form a one-dimensional group G_D with parameter s . The part $\varphi_D(s)$ form themselves a group $G_{D\varphi}$ that is a subgroup of $\text{Diff}\mathcal{M}$. The dilatations act on the physical phase space:

$$\left(\varphi_D(s), \omega(\varphi_D(s))\right) : (\eta, M, w) \mapsto (\eta, e^s M, e^s w) .$$

Finally, dilatations act on the boundary

$$U_\infty \mapsto e^s U_\infty , \quad V_\infty \mapsto e^s V_\infty , \quad T_0 \mapsto e^s T_0 .$$

The time shifts and dilatations generate a group. Each element of the group can be obtained in a unique way as a dilatation followed by a time shift. Let us write it in the form $\left(\varphi_H(t), \varphi_D(s), \omega(\varphi_D(s))\right)$. If the action of the constituent maps on, say, the parameter w is taken into account, the following composition law can be found

$$\begin{aligned} \left(\varphi_H(t_1), \varphi_D(s_1), \omega(\varphi_D(s_1))\right) \times \\ \left(\varphi_H(t_2), \varphi_D(s_2), \omega(\varphi_D(s_2))\right) = \\ \left(\varphi_H(t_1 + e^{s_1} t_2), \varphi_D(s_1 + s_2), \omega(\varphi_D(s_1 + s_2))\right) . \end{aligned}$$

It shows that the group has the structure of semi-direct product of the multiplicative \mathbb{R}_+ with the additive \mathbb{R} abelian groups ($t \in \mathbb{R}$, $e^s \in \mathbb{R}_+$). This group is usually called *affine group* \mathcal{A} on \mathbb{R} . The time shifts form a normal subgroup of \mathcal{A} : they play the role of the abelian factor in the semi-direct product.

Time reversal The two solutions $\Psi_\eta(M, w; U, V)$ and $\Psi_{-\eta}(M, -w; U, V)$ are isometric. The isometry is implemented by the diffeomorphism φ_I defined by

$$\varphi_I : (U, V) \mapsto (-V, -U) .$$

It is well-defined everywhere in \mathcal{M} and independent of η , M and w . Its action on the boundary is

$$U_\infty \mapsto -V_\infty , \quad V_\infty \mapsto -U_\infty , \quad T_0 \mapsto -T_0 ,$$

and that on the physical phase space is

$$I : (\eta, M, w) \mapsto (-\eta, M, -w) .$$

The full group of symmetries generated by all three kinds of maps is denoted by G_σ . The index σ is to remind that the action of the group on \mathcal{M} has been obtained with the help of our solution list that represents the gauge fixing σ ; a different gauge fixing would lead to different action on \mathcal{M} . However, the action of G_σ on the physical phase space as well as some aspects of its action on the boundary are gauge independent. So is the group structure of G_σ , which is that of the affine group \mathcal{A}_I with inversion.

The action of G_σ on the boundary $\partial\mathcal{M}$ defines a group of boundary transformations that will be denoted by G_b . The group G_b consists of pure diffeomorphisms (not polluted by any CCD's). Each element of G_σ determines a unique element of G_b and this map between G_σ and G_b is an isomorphism because G_σ acts truly on the boundary.

There is one point about the structure of G_b that will be useful later: the subgroup generated by time shifts and the time reversal is invariant (normal). Hence, there is a homeomorphism π_D of G_b onto the factor group, which is $G_{D\varphi}$, $\pi_D : G_b \mapsto G_{D\varphi}$. This means that the amount of dilatation hidden in any element of G_b is well-defined.

Decomposition of $\text{Diff}\mathcal{M}$ Let us choose a subgroup G of $\text{Diff}\mathcal{M}$ according to the following rules. The diffeomorphism φ is an element of G if

1. the map φ has a differentiable extension to the boundary of \mathcal{M} . Hence, it defines a map $\varphi_b : \partial\mathcal{M} \mapsto \partial\mathcal{M}$,
2. The action of φ on the boundary coincides with that of an element of the symmetry group G_σ , i.e., $\varphi_b \in G_b$.

The group G is a proper subgroup of $\text{Diff}\mathcal{M}$. It is the only part of $\text{Diff}\mathcal{M}$ that is interesting for the physics of the system. It is easy to check that the map $\pi_b : G \mapsto G_b$, the existence of which follows from the above definition, is a homeomorphism (preserves multiplication). The kernel G_0 of π_b is a set of elements of G that is mapped by π_b to the identity of G_b . Such a kernel must be a normal subgroup of G . The group G_0 is the *gauge group* of the system.

The group G itself needs a correction by CCD's at those elements that can be said to contain dilatation. The amount of dilatation in $\varphi \in G$ is obviously given by $\pi_D\pi_b\varphi$. Hence, the pairs $(\varphi, \omega(\pi_D\pi_b\varphi))$ are *symmetries* of our system. They form a group with the same structure as G because CCD's commute with diffeomorphisms. Let us denote the group by G_c (c for corrected).

The action of the symmetry group G_c on the boundary has a gauge-invariant aspect. This can be revealed after attaching the scri to the solutions. Consider, for example, the spacetime $(\mathcal{M}, \Psi_+(M, w; U, V))$. The $\mathcal{I}^+(+1, M, w)$ of this spacetime is one-dimensional and a coordinate \tilde{U} can be chosen along it so that the difference

$\tilde{U}_2 - \tilde{U}_1$ gives the interval of retarded time along $\mathcal{I}^+(+1, M, w)$. In the spherically symmetric case, such an interval is uniquely determined. However, \tilde{U} clearly contains more information than just that about intervals: an origin of \tilde{U} has also been chosen. This origin is the only non-trivial remainder, in the spherically symmetric case, of what can be called *asymptotic frame*. In an analogous way, an inertial coordinate system in Minkowski spacetime contains not only information about Minkowski interval (metric), but also that on the underlying inertial frame.

Let us attach $\mathcal{I}^+(+1, M, w)$ so that

$$\tilde{U} = U_\infty . \quad (24)$$

This is, in any case, possible because U_∞ is a retarded time along $\partial_+\mathcal{M}$ for the spacetime $(\mathcal{M}, \Psi_+(M, w; U, V))$ so that the intervals match. However, Eq. (24) represents also a judicious choice of origins, that is, asymptotic frames, for each solution $\Psi_+(M, w; U, V)$. The idea behind the choice is that w keeps the role of the retarded arrival time of the out-going shells also with respect to $\mathcal{I}^+(+1, M, w)$. We shall see later how this choice of frame of asymptotic reference influences the form of the asymptotic action of the symmetry group and the calculation of the symplectic form.

At $\partial_-\mathcal{M}$, where $U \rightarrow -\infty$ and $V \in (w + 4M, \infty)$, we must recover the Schwarzschild advanced time \tilde{V} as a function of V in order to attach $\mathcal{I}^-(+1, M, w)$. An easy calculation yields:

$$\tilde{V} = V_\infty + 4M \ln \left(\frac{-w + V_\infty}{4M} - 1 \right) , \quad V_\infty = w + 4M\kappa \left[\exp \left(\frac{\tilde{V}}{4M} \right) \right] . \quad (25)$$

As $V \in (w + 4M, \infty)$, \tilde{V} runs through \mathbb{R} . The advanced time interval is just $\tilde{V}_2 - \tilde{V}_1$. Eq. (25) replaces (24) and determines the attachment of $\mathcal{I}^-(+1, M, w)$ to the spacetime $(\mathcal{M}, \Psi_+(M, w; U, V))$. Observe that $\mathcal{I}^-(+1, M, w)$ takes only a part of $\partial_-\mathcal{M}$.

Finally, $\mathcal{I}^0(+1, M, w)$ is a “stretched” space-like infinity i^0 : it has the structure of \mathbb{R} and the coordinate T_∞ on it can be defined as limit point of a constant Schwarzschild time $T = T_\infty$ surfaces. Such surfaces are defined in the coordinates U and V by

$$\frac{U + \tilde{V}(V)}{2} = T_\infty .$$

We obtain from Eq. (25):

$$\frac{U}{2} + \frac{V}{2} + 2M \ln \left(\frac{-w + V}{4M} - 1 \right) = \text{const}$$

as $U \rightarrow -\infty$ and $V \rightarrow \infty$. The surfaces are defined by the functions $(U(R), V(R))$:

$$U = -R - 2M \ln \left(\frac{R}{2M} - 1 \right) + T_\infty \quad (26)$$

and

$$V = w + 4M\kappa \left[\exp \left(\frac{T_\infty + R}{4M} \right) \sqrt{\frac{R}{2M} - 1} \right]. \quad (27)$$

The Schwarzschild time interval is the difference $T_{\infty 2} - T_{\infty 1}$. Again Eqs. (25)–(27) not only imply that the intervals as calculated from \tilde{V} or T_∞ match with intervals calculated from the geometry of $(\mathcal{M}, \Psi_+(M, w; U, V))$. They also represent choices of origins, i.e., asymptotic frames, at $\mathcal{I}^-(+1, M, w)$ and $\mathcal{I}^0(+1, M, w)$, for each solution $\Psi_+(M, w; U, V)$.

The relation between the origins of the three spaces \mathcal{I}^+ , \mathcal{I}^- and \mathcal{I}^0 corresponding to *one* Schwarzschild spacetime (i.e., for a fixed M) can be set by a geometric convention. For example, the three surfaces $\tilde{U} = 0$, $\tilde{V} = 0$ and $T_\infty = 0$ (if the coordinates \tilde{U} , \tilde{V} and T_∞ are extended into the spacetime as null or maximal surfaces) can be required to intersect all at a sphere of radius $R = 2M\kappa(1)$. This is the convention we use for each η , M and w . In this way, the retarded arrival time w can also be calculated by a standardized way from the description of the dynamics of the shell in an extended frame associated with \mathcal{I}^- and \mathcal{I}^0 . This is important for the action of time shifts as well as for the calculation of the Liouville form. The dependence of the geometric convention on the parameters η and M influences the form of asymptotic action of dilatations and the time reversal.

The attachment of $\mathcal{I}^+(-1, M, w)$, $\mathcal{I}^-(-1, M, w)$ and $\mathcal{I}^0(-1, M, w)$ to the spacetimes $(\mathcal{M}, \Psi_-(M, w; U, V))$ is entirely analogous and can be skipped.

We observe that the scries have not, in general, a fixed position in the background manifold \mathcal{M} . This is due to the gauge fixing σ . Any gauge fixing implies also a definition of points (events) of \mathcal{M} by some of their geometrical properties. For example, at the point (U, V) , two light-like surfaces meet that have a particular geometrical meaning. The point at scri can also be defined by some geometrical properties. The latter definition does not coincide with a limit of the former in our case. This does not lead to any problems, if one does not try to use the definition of points inside \mathcal{M} in an improper way: it is not gauge invariant and has not much physical meaning.

On the other hand, we can use the gauge fixing to calculate a lot of useful, gauge invariant results. For example, we can determine how the group G_c acts on the scri. Clearly, G_c acts there only via G_b so that, for example, G_0 must act trivially, leaving all points of scri invariant. For general $\varphi \in G$, a simple calculation reveals

the following pattern. Let the diffeomorphism φ acts on the physical phase space as follows

$$\varphi : (\eta, M, w) \mapsto (\eta, M', w') ,$$

that is, φ does not contain any time reversal. Then

$$\varphi_b : \mathcal{I}^+(\eta, M, w) \mapsto \mathcal{I}^+(\eta, M', w') \quad (28)$$

and the map is a *bijection* between the two scries. The same claim holds for \mathcal{I}^- and \mathcal{I}^0 . For example, the scri $\mathcal{I}^-(+1, M, w)$ has a past end point at $\partial_- \mathcal{M}$ with the value V_{past} of the coordinate V , $V_{\text{past}} = w + 4M$. The time shift $\varphi_H(t)$ sends V to $V + t$, hence $V_{\text{past}} \mapsto V_{\text{past}} + t$, and this is the past end point of $\mathcal{I}^-(+1, M, w + t)$.

Similarly, the time reversal

$$I : \mathcal{I}^+(\eta, M, w) \mapsto \mathcal{I}^-(-\eta, M, -w) , \quad (29)$$

etc., defines a bijection between the two scries.

Actually, we are going to identify all \mathcal{I}^+ 's considering the points with the same value of the coordinate \tilde{U} as equal. This gives a common \mathcal{I}^+ for all solutions. Similarly, common \mathcal{I}^- and \mathcal{I}^0 can be defined. This structure is very important for the interpretation of the theory. We assume that observers are living at this common scri and are using the common frames (origins of time).

The definition of the common scri given above, together with the equations such as (28) and (29), imply that the group G_c acts on the common scri. This action is not difficult to calculate. The result is:

$$\begin{aligned} \varphi_H(t) & : (\tilde{U}, \tilde{V}, T_\infty) \mapsto (\tilde{U} + t, \tilde{V} + t, T_\infty + t) , \\ \varphi_D(s) & : (\tilde{U}, \tilde{V}, T_\infty) \mapsto (\tilde{U}e^s, \tilde{V}e^s, T_\infty e^s) , \\ \varphi_H(t) & : (\tilde{U}, \tilde{V}, T_\infty) \mapsto (-\tilde{V}, -\tilde{U}, -T_\infty) . \end{aligned}$$

The fact that the action of the time shift by t shifts all coordinates \tilde{U} , \tilde{V} and T_∞ by the same amount is the consequence of the judicious choice of origins. Similarly, the homogeneous action of the dilatations is due to correlations of our choices of frames for all different values of the parameter M .

Clearly, the group G_c acts transitively on Γ . If the symplectic structure of the physical phase space Γ were known, the functions generating the infinitesimal transformations of the group could be found. Then the dynamics based on the time shifts could be constructed.

3.2 Canonical theory

The variables η , M and w can play the role of Dirac observables, but we do not know what are their Poisson brackets. In the present section, we calculate the brackets. We start from a Hamiltonian action principle for null shells and reduce it to physical phase space applying the technique invented by Kuchař.

3.2.1 The action

As a Hamiltonian action principle that implies the dynamics of our system, we take the action Eq. (2.6) of [30] (see also [31]). Let us briefly summarize the relevant formulae. The spherically symmetric metric is written in the form:

$$ds^2 = -N^2 d\tau^2 + \Lambda^2 (d\rho + N^\rho d\tau)^2 + R^2 d\Omega^2 , \quad (30)$$

the shell is described by its radial coordinate $\rho = \mathbf{r}$ and its conjugate momentum \mathbf{p} . The action reads

$$S_0 = \int d\tau \left[\mathbf{p} \dot{\mathbf{r}} + \int d\rho (P_\Lambda \dot{\Lambda} + P_R \dot{R} - H_0) \right] , \quad (31)$$

and the Hamiltonian is

$$H_0 = N\mathcal{H} + N^\rho \mathcal{H}_\rho + N_\infty E_\infty ,$$

where N and N^ρ are the lapse and shift functions, \mathcal{H} and \mathcal{H}_ρ are the constraints,

$$\begin{aligned} \mathcal{H} = & \text{G} \left(\frac{\Lambda P_\Lambda^2}{2R^2} - \frac{P_\Lambda P_R}{R} \right) + \frac{1}{\text{G}} \left(\frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2} \right) \\ & + \frac{\eta \mathbf{p}}{\Lambda} \delta(\rho - \mathbf{r}) , \end{aligned} \quad (32)$$

$$\mathcal{H}_\rho = P_R R' - P'_\Lambda \Lambda - \mathbf{p} \delta(\rho - \mathbf{r}) , \quad (33)$$

and the prime or dot denote the derivatives with respect to ρ or τ . The term $N_\infty E_\infty$ is the remainder of $\mathcal{H}_\infty[\vec{N}_\infty]$ (see Sec. 2.2) in the case of rotational symmetry. $N_\infty := \lim_{\rho \rightarrow \infty} N^\rho(\rho)$ and E_∞ is the ADM mass (see [30]). In the Schwarzschild spacetime with mass parameter M and asymptotic Schwarzschild time T_∞ , it holds that

$$N_\infty E_\infty = \frac{1}{\text{G}} M \dot{T}_\infty .$$

The term can then be transferred from the Hamiltonian to the part of action containing time derivatives—the so-called *Liouville form*, see [32].

The Hamiltonian constraint function \mathcal{H} is written in a way that differs from [30] and [6] in that its dependence on the Newton constant G becomes visible.

This is necessary in order that the scaling properties⁶ of \mathcal{H} are manifest, $\mathcal{H} \mapsto e^{-s}\mathcal{H}$. Observe that the rescaling of \mathcal{H} cannot be implemented by a canonical transformation: all function that might be tried to generate such rescaling had vanishing Poisson brackets with G . This does not mean, however, that vacuum Einstein equations are not invariant under dilatation (cf. [33]). In our case, the geometric sector alone (see Sec. 3.1) as well as the matter sector alone, are invariant under dilatation, but the whole theory is not: the relation between geometry and matter involves the Newton constant.

The “volume” variables Λ , R , P_Λ , P_R , N and N^ρ are the same as in [30] and [6]. The meaning of the variables Λ , R , N and N^ρ can be inferred from the spacetime metric (30). The momenta conjugate to the configuration variables Λ and R can be calculated from the action S_0 by varying it with respect to P_Λ and P_R :

$$P_\Lambda = -\frac{R}{GN}(\dot{R} - N^\rho R') , \quad (34)$$

and

$$P_R = -\frac{\Lambda}{GN}(\dot{R} - N^\rho R') - \frac{R}{GN}[\dot{\Lambda} - (N^\rho \Lambda)'] . \quad (35)$$

Some differentiability conditions at the shell are important in order that proper equations of motion are obtained by varying the action. One can assume as in [30] that the gravitational variables are smooth functions of ρ , with the exception that N' , $(N^\rho)'$, Λ' , R' , P_Λ and P_R may have finite discontinuities at isolated values of ρ . The coordinate loci of the discontinuities are smooth functions of τ for each shell. This follows from (i) the conditions at shell points which are the same as in [30] and (ii) from the corresponding choice of foliation: the metric with respect to coordinates τ and ρ may be piecewise smooth and everywhere continuous.

3.2.2 The Liouville form at the constraint surface

Our aim is to calculate the Poisson brackets between Dirac observables such as M and w . We can employ the property of the pull-back $\Theta_{\mathcal{C}}$ of the Liouville form Θ to the constraint surface \mathcal{C} that it depends only on the Dirac observables as it has been explained in Sec. 2 (see also [9], [3] and [4]). Its external differential then defines the symplectic form of the physical phase space, which determine the brackets. Observe further that the pull-back $\Theta_{\mathcal{C}}$, if integrated over τ , gives the action of the reduced system. Indeed, if one solves the constraints, only the Liouville form remains from the action S_0 .

⁶To reveal the scaling behaviour, units must be chosen so that $\hbar = c = 1$, but $G \neq 1$!

Thus, we have to transform the Liouville form to the variables M , w and a set of observable-dependent embeddings; these variables form a coordinate system on the constraint surface for each case $\eta = +1$ and $\eta = -1$.

An important point is to specify the family of embeddings that will be used. The embeddings are given by

$$U = U(o, \rho), \quad V = V(o, \rho) ,$$

where U and V are the coordinates defined in Sec. 2.1. These functions have to satisfy several conditions.

1. As Σ is space-like, U and V are null and increasing towards the future, we must have $U' < 0$ and $V' > 0$ everywhere.
2. At the regular centre, the four-metric is flat and the three-metric is to be smooth. This implies, for all o 's, that $U'(o, 0) = -V'(o, 0)$ in addition to the condition $U(o, 0) = V(o, 0)$. This follows from $T'(U(o, 0), V(o, 0)) = 0$ and means that Σ must run parallel to $T = \text{const}$ in order to avoid conical singularities. Here, $T = (U + V)/2$ is the time of the inertial system at the centre.
3. At the space-like infinity, the four-metric is the Schwarzschild metric. We require that the embedding approaches the Schwarzschild-time-constant surfaces $T = \text{const}$, and that ρ becomes the Schwarzschild curvature coordinate R asymptotically. More precisely, the behaviour of the Schwarzschild coordinates T and R along each embedding $U(o, \rho), V(o, \rho)$ must satisfy

$$T(\rho) = T_\infty + O(\rho^{-1}), \tag{36}$$

$$R(\rho) = \rho + O(\rho^{-1}) . \tag{37}$$

The asymptotic coordinate T_∞ is a gauge-invariant quantity and it possesses the status of an observable. It follows then from Eqs. (26) and (27) that the embeddings must depend on η , M and w .

4. At the shell ($\rho = \mathbf{r}$) we require $U(o, \rho)$ and $V(o, \rho)$ to be C^∞ functions of ρ . In fact, as the four-metric is continuous in the coordinates U and V , but not smooth, only the C^1 -part of this condition is gauge invariant. Jumps in all higher derivatives are gauge dependent, but the condition will simplify equations without influencing results.

The Liouville form of the action (14) can be written as follows:

$$\Theta = \mathbf{p}\dot{\mathbf{r}} - \frac{1}{G}M\dot{T}_\infty + \int_0^\infty d\rho (P_\Lambda \dot{\Lambda} + P_R \dot{R}) . \tag{38}$$

We can now start to transform (38) into Kuchař variables. It is advantageous to let first the double-null coordinates U and V arbitrary and the Dirac observables o^i , $i = 1, \dots, 2n$ unspecified. We just need to know that the metric and the embeddings depends on o^i :

$$A = A(U, V; o) , \quad R = R(U, V; o) , \quad U = U(o, \rho) , \quad V = V(o, \rho) .$$

For any double-null gauge, the equations

$$4RR_{,UV} + 4R_{,U}R_{,V} + A = 0 , \quad (39)$$

$$AR_{,UU} - A_{,U}R_{,U} = 0 , \quad (40)$$

$$AR_{,VV} - A_{,V}R_{,V} = 0 \quad (41)$$

represent the condition that the transformation is performed at the constraint surface Γ (cf. [6], Eqs. (32)–(34)).

The transformation formulae for Λ , R , N and N^ρ can be read off the metric:

$$R = R(U(o, \rho), V(o, \rho), o) , \quad (42)$$

$$\Lambda = \sqrt{-AU'V'} , \quad (43)$$

$$\mathcal{N} = -\frac{\dot{U}V' - \dot{V}U'}{2U'V'}\sqrt{-AU'V'} , \quad (44)$$

$$\mathcal{N}^\rho = \frac{\dot{U}V' + \dot{V}U'}{2U'V'} . \quad (45)$$

The transformation of the momenta P_R and P_Λ are obtained from their definitions (34) and (35), into which Eqs. (42)–(45) are substituted:

$$GP_\Lambda = \frac{R}{\sqrt{-AU'V'}} (R_{,U}U' - R_{,V}V') , \quad (46)$$

$$GP_R = R_{,U}U' - R_{,V}V' + \frac{RA_{,U}}{2A}U' - \frac{RA_{,V}}{2A}V' + \frac{RU''}{2U'} - \frac{RV''}{2V'} . \quad (47)$$

The transformation of the shell variable \mathbf{r} follows from the obvious relations

$$U(o, \mathbf{r}) = w , \quad V(o, \mathbf{r}) = w ,$$

each valid in one of the cases $\eta = +1$ or $\eta = -1$. Finally, the value of the momentum \mathbf{p} at the constraint surface can be found if the differentiability properties of the volume functions at the shell are used and the coefficients at the δ -functions in the constraints (32) and (33) are compared (for details see [6]):

$$\mathbf{p} = -\eta R \Delta_{\mathbf{r}}(R') = -G \Lambda \Delta_{\mathbf{r}}(P_\Lambda) ,$$

where $\Delta_{\mathbf{r}}(f)$ denotes a jump in the function f across the shell at the point $\rho = \mathbf{r}$.

The form (38) can be divided into a boundary part (the first two terms on the right-hand side) and the volume parts. Each volume part is associated with a particular component of the space cut out by the shell; it has the form

$$\int_a^b d\rho (P_\Lambda \dot{\Lambda} + P_R \dot{R}) , \quad (48)$$

where a and b are values of the coordinate ρ at the boundary of the volume. There are only two cases, $a = 0, b = \mathbf{r}$ and $a = \mathbf{r}, b = \infty$.

Since the pull-back of the Liouville form cannot depend on the volume variables $U(o, \rho)$ and $V(o, \rho)$, its volume part must be, after the transformation, given by

$$\Theta_a^b|_c = \int_a^b d\rho [(f\dot{U} + g\dot{V} + h_i \dot{o}^i)' + \dot{\varphi}] , \quad (49)$$

cf. [6]. Comparing the coefficients at the dotted and primed quantities in Eqs. (48) and (49) leads to partial differential equations for the functions f, g and h_i [6] and one can show that Eqs. (39)–(41) are the integrability conditions for these partial differential equations.

In such a way, Θ_c can be transformed to a sum of boundary terms. These terms can be calculated and simplified if the transformation equations for the shell variables and the properties of the embeddings at the boundary points are used. It is a lengthy and rather technical calculation that is not very interesting for us now; it has been described in detail in [6] and, especially, [28]. We skip it and quote only the final results:

$$\eta = +1 : \quad \Theta|_c = -\frac{1}{G} M \dot{w} ; \quad \eta = -1 : \quad \Theta|_c = -\frac{1}{G} M \dot{w} . \quad (50)$$

Only one aspect of the calculation deserves mentioning. This is the cancellation of the term $-(M/G)\dot{T}_\infty$ in the Liouville form (38) by a term coming from the volume part (48). The cancellation is again a result of our judicious choice of asymptotic frames as described in Sec. 3.1: the variable T_∞ enters the calculation via Eqs. (26) and (27) that are implied by the choice.

At this stage, it is advantageous to introduce new variables for the description of the Dirac observables, that is, for the matter sector in our case: let us define, for $\eta = +1$:

$$p_u := -\frac{M}{G} , \quad u := w ,$$

and for $\eta = -1$,

$$p_v := -\frac{M}{G} , \quad v := w .$$

Observe that the momenta p_u and p_v have the meaning of negative energy and scale in accord with this meaning as e^{-s} . The rescaling of p 's, together with the rescaling by e^s of u and v , can be implemented canonically.

The desired Poisson brackets can be written down immediately:

$$\{u, p_u\} = 1, \quad \{v, p_v\} = 1.$$

the generator of the infinitesimal time shift is $p_u dt$ or $p_v dt$ and that of dilatation is $p_u u ds$ or $p_v v ds$.

In [28], a general method of integration of the differential equations for the functions f , g and h_i has been developed and a formula generalizing Eq. (50) to a system containing any number of in-going and any number out-going shell and the due shell intersections (Kouletsis formula) has been written down.

The reduced action that contains only the physical degrees of freedom is, if $\eta = +1$,

$$S_+ = \int d\tau p_u \dot{u}, \quad (51)$$

and if $\eta = -1$,

$$S_- = \int d\tau p_v \dot{v}. \quad (52)$$

These actions imply the equations of motion

$$p_u = \text{const}, \quad v = \text{const}$$

or

$$p_v = \text{const}, \quad u = \text{const},$$

as they are to be.

3.2.3 Merging in- and out-going dynamics

In this subsection, the two reduced action principles for the out-going and in-going dynamics of the shell will be recognized as a result of reduction of a single merger action. The merger action gives a complete account of all motions of the shell on the background manifold \mathcal{M} . The merging is a step of certain significance for our quantum theory.

Consider the action (51) for out-going shells. The variables p_u and u do not contain the full information about where the shell is in \mathcal{M} . The function $u(\tau)$ is the value of only one coordinate U of the shell for the value τ of the time parameter.

For a complete description, $v(\tau)$ had to be added. For a solution trajectory, $u(\tau)$ is constant, but $v(\tau)$ can be arbitrary: it depends on the choice of the parameter τ and so it can be considered as a gauge variable in a reparametrization-invariant formalism. Hence, a valid extension of the action (51) is

$$\bar{S}_+ = \int d\tau (p_u \dot{u} + p_v \dot{v} - n_+ p_v) ,$$

where n_+ is a Lagrange multiplier that enforces the vanishing of the momentum p_v conjugate to the gauge variable v . The meaning of n_+ transpires from the equation

$$\dot{v} = n_+$$

obtained by varying \bar{S}_+ with respect to p_v . Thus, n_+ measures the rate of τ with respect to the parameter $2t$: for any out-going shell, we have

$$t = \frac{1}{2}(u + v(\tau)) ,$$

where t is the time of the inertial frame defined by the regular centre $R = 0$ and, at the same time, it coincides with the parameter of the time-shift group.

We can define another Lagrangian multiplier n by

$$n_+ = n p_u ,$$

because p_u is always non-zero in the case $\eta = +1$. The multiplier n measures the rate of τ with respect to the “physical parameter”: the corresponding tangent vector to particle trajectory coincides with the contravariant four-momentum of the particle.

An analogous extension \bar{S}_- of S_- , Eq. (52) is

$$\bar{S}_- = \int d\tau (p_u \dot{u} + p_v \dot{v} - n_- p_u) ,$$

and we can switch to n defined by $n_- = n p_v$ now.

The merger action is

$$\bar{S} = \int d\tau (p_u \dot{u} + p_v \dot{v} - n p_u p_v) . \quad (53)$$

It contains both cases as the two possible solutions of the constraint

$$p_u p_v = 0 :$$

if $p_v = 0$, we obtain the case $\eta = +1$, and if $p_u = 0$, $\eta = -1$.

The two variables u and v determine the position of the shell in \mathcal{M} uniquely. The generator of the time translation is $p_u + p_v$, the value of which is the negative

of the total energy $E = M/G$ of the system, and the generator of the dilatation is $p_u u + p_v v$.

The new phase space has non-trivial boundaries:

$$p_u \leq 0, \quad p_v \leq 0, \quad (54)$$

$$\frac{-u + v}{2} > 0, \quad (55)$$

$$p_v = 0, \quad U \in (-\infty, u), \quad V > u - 4p_u \kappa \left(-\exp \frac{u - U}{4p_u} \right), \quad (56)$$

and

$$p_u = 0, \quad V \in (v, \infty), \quad U < v + 4p_v \kappa \left(-\exp \frac{V - v}{4p_v} \right). \quad (57)$$

The boundaries defined by inequalities (56) and (57) are due to the singularity. Eqs. (56) and (57) limit the values of U and V that can be used for embeddings.

The two dynamical systems defined by the actions (31) and (53) are equivalent: each maximal dynamical trajectory of the first, if transformed to the new variables, gives a maximal dynamical trajectory of the second and vice versa.

The variables u, v, p_u and p_v span the effective phase space of the shell. They contain all true degrees of freedom of the system. One can observe that the action (53) coincides with the action for free motion of a zero-rest-mass spherically symmetric (light-like) shell in flat spacetime. Such a dynamics is complete because there is no geometric singularity at the value zero of the radius of the shell, $(-u + v)/2$, and this point can be considered as a harmless caustic so that the light can re-expand after passing through it. The dynamics of the physical degrees of freedom by itself is, therefore, regular.

It might seem possible to extend the phase space of the gravitating shell, too, in the same way so that the in-going and the out-going sectors are merged together into one bouncing solution. However, such a formal extension of the dynamics (53) is not adequate. The physical meaning of any solution written in terms of new variables is given by measurable quantities of geometrical or physical nature, which include now also the curvature of spacetime. These observables must be expressed as functions on the phase space. They can of course be transformed between the phase spaces of the two systems (31) and (53). They cannot be left out from any complete description of a system, though they are often included only tacitly: an action alone does not define a system. This holds just as well for the action (31) as for (53).

Let us consider these observables. Eqs. (10)–(13) and (15)–(18) can be used to show that the curvature of the solution spacetime diverges at the boundary defined by Eq. (56) for $p_v = 0$ and by Eq. (57) for $p_u = 0$. It follows that the observable quantities at and near the “caustic” are badly singular and that there is no sensible extension of the dynamics defined by action (53) to it, let alone through it. This confirms the more or less obvious fact that no measurable property (such as the singularity) can be changed by a transformation of variables.

The action for the null dust shell is now written in a form which can be taken as the starting point for quantization. Surprisingly, it will turn out that a quantum theory can be constructed so that it is singularity-free. This will be shown in the next section.

4 Quantum theory

In this section, we shall construct a quantum theory of our model. Of course, there is no unique construction of a quantum theory for any given classical model. We just show that there is a singularity-free, unitary quantum mechanics of the shell. The account in this section follows [34] adding a few new ideas.

4.1 Group quantization

To quantize the system defined by the action (53), we apply the so-called group-theoretical quantization method [35]. There are three reasons for this choice. First, the method as modified for the generally covariant systems by Rovelli [36] (see also [37] and [38]) is based on the algebra of Dirac observables of the system; dependent degrees of freedom don’t influence the definition of Hilbert space. Second, the group method has, in fact, been invented to cope with restrictions such as Eqs. (54) and (55). By and large, one has to choose a set of observables that form a Lie algebra; this algebra has to generate a group of symplectomorphisms that has to act transitively in the phase space respecting all boundaries. In this way, the information about the boundaries is built in the quantum mechanics. Finally, the method automatically leads to self-adjoint operators representing all observables. In particular, a self-adjoint extension of the Hamiltonian is obtained in this way, and this is the reason that the dynamics is unitary.

To begin with, we have to find a complete set of Dirac observables. Let us choose the functions $p_u, p_v, D_u := up_u$ and $D_v := vp_v$. Observe that u alone is constant only along out-going shell trajectories ($p_u \neq 0$), and v only along in-going ones ($p_v \neq 0$), but up_u and vp_v are always constant. The only non vanishing Poisson brackets are

$$\{D_u, p_u\} = p_u, \quad \{D_v, p_v\} = p_v.$$

This Lie algebra generates a group Q_2 of symplectic transformations of the phase space that preserve the boundaries $p_u = 0$ and $p_v = 0$. Q_2 is the Cartesian product of two copies of the affine group \mathcal{A} on \mathbb{R} .

The group \mathcal{A} generated by p_u and D_u has three irreducible unitary representations. In the first one, the spectrum of the operator \hat{p}_u is $[0, \infty)$, in the second, \hat{p}_u is the zero operator, and in the third, the spectrum is $(-\infty, 0]$, see Ref. [39]. Thus, we must choose the third representation; this can be described as follows (details are given in Ref. [39]).

The Hilbert space is constructed from complex functions $\psi_u(p)$ of $p \in [0, \infty)$; the scalar product is defined by

$$(\psi_u, \phi_u) := \int_0^\infty \frac{dp}{p} \psi_u^*(p) \phi_u(p),$$

and the action of the generators \hat{p}_u and \hat{D}_u on smooth functions is

$$(\hat{p}_u \psi_u)(p) = -p \psi_u(p), \quad (\hat{D}_u \psi_u)(p) = -ip \frac{d\psi_u(p)}{dp}.$$

Similarly, the group generated by p_v and D_v is represented on functions $\psi_v(p)$; the group Q_2 can, therefore, be represented on pairs $(\psi_u(p), \psi_v(p))$ of functions:

$$\begin{aligned} \hat{p}_u(\psi_u(p), \psi_v(p)) &= (-p \psi_u(p), 0), \\ \hat{p}_v(\psi_u(p), \psi_v(p)) &= (0, -p \psi_v(p)), \\ \hat{D}_u(\psi_u(p), \psi_v(p)) &= \left(-ip \frac{d\psi_u(p)}{dp}, 0\right), \\ \hat{D}_v(\psi_u(p), \psi_v(p)) &= \left(0, -ip \frac{d\psi_v(p)}{dp}\right), \end{aligned}$$

This choice guarantees that the Casimir operator $\hat{p}_u \hat{p}_v$ is the zero operator on this Hilbert space, and so the constraint is satisfied. The scalar product is

$$\left((\psi_u(p), \psi_v(p)), (\phi_u(p), \phi_v(p)) \right) = \int_0^\infty \frac{dp}{p} (\psi_u^*(p) \phi_u(p) + \psi_v^*(p) \phi_v(p)).$$

Let us call this representation \mathcal{R}_2 . Observe that this representation of the group $\mathcal{A} \times \mathcal{A}$ is not irreducible. It is, however, irreducible for the group extended by the time reversal because its representative swaps the invariant subspaces. The representation \mathcal{R}_2 has a well-defined meaning as a quantum theory of our system. In it, the out-going shells are independent of the in-going ones. The dynamics of each kind of shells is complete for itself. One could study this dynamics by defining a position operator $\hat{r}(t)$ in a natural way similar to what will be done later and

one would find that the in-going shells simply proceed through $r = 0$ into negative values of r . Analogous holds, in the time reversed order, for the out-going shells.

Handling the inequality (55) is facilitated by the canonical transformation:

$$t = (u + v)/2, \quad r = (-u + v)/2, \quad (58)$$

$$p_t = p_u + p_v, \quad p_r = -p_u + p_v. \quad (59)$$

The constraint function then becomes $p_u p_v = (p_t^2 - p_r^2)/4$.

The function $p_t \delta t$ generates, via Poisson brackets, the infinitesimal time shift in (t, r) -space, $t \mapsto t + \delta t$, $r \mapsto r$, and $p_r \delta r$ generates an r -shift, $t \mapsto t$, $r \mapsto r + \delta r$. We introduce also the observables

$$D := D_u + D_v = t p_t + r p_r ,$$

and

$$J := -D_u + D_v = r p_t + t p_r .$$

The function $D \delta s$ generates a dilatation in the (t, r) -space, $t \mapsto t + t \delta s$, $r \mapsto r + r \delta s$, and $J \delta v$ generates a boost, $t \mapsto t + r \delta v$, $r \mapsto r + t \delta v$. What do these transformations with our half-plane $r > 0$? The transformations generated by p_t and D preserve the boundary while those by p_r and J do not. Hence, only the subgroup Q_1 of Q_2 generated by p_t and D respects the inequality (55).

The representation \mathcal{R}_2 of Q_2 can be decomposed into the direct sum of two equivalent representations of Q_1 . This representation of Q_1 will be denoted by \mathcal{R}_1 and it will serve as our *definitive* quantum mechanics. Let us stress that this quantum mechanics does not describe a subsystem of the physical system described with the representation \mathcal{R}_2 . The representation \mathcal{R}_1 can be described as follows: The states are determined by complex functions $\varphi(p)$ on \mathbb{R}_+ ; the scalar product (φ, ψ) is

$$(\varphi, \psi) = \int_0^\infty \frac{dp}{p} \varphi^*(p) \psi(p);$$

let us denote the corresponding Hilbert space by \mathcal{K} . The representatives of the above algebra are

$$\begin{aligned} (\hat{p}_t \varphi)(p) &= -p \varphi(p), \\ (\hat{p}_r^2 \varphi)(p) &= p^2 \varphi(p), \\ (\hat{D} \varphi)(p) &= -ip \frac{d\varphi(p)}{dp}, \\ (\hat{J}^2 \varphi)(p) &= -p \frac{d\varphi(p)}{dp} - p^2 \frac{d^2 \varphi(p)}{dp^2}. \end{aligned}$$

There are also well-defined operators for p_r^2 and J^2 because, in the representation \mathcal{R}_2 , the identities $\hat{p}_r^2 = \hat{p}_t^2$ and $\hat{J}^2 = \hat{D}^2$ hold.

An important observation is that \mathcal{R}_1 is a representation of the group of asymptotic symmetries as described in Sec. 3.1. Another important observation is that the quantum mechanics \mathcal{R}_2 describes two decoupled degrees of freedom, namely the in- and out-going shells, while \mathcal{R}_1 describes a system with a single degree of freedom: the in- and out-going motions have been coupled into one motion. Let us study, what is this motion.

First, we have to construct a time evolution. For that, we use the time shift symmetry generated by \hat{p}_t . The operator $-\hat{p}_t$ has the meaning of the total energy $E = M/G$ of the system. We observe that it is a self-adjoint operator with a positive spectrum and that it is diagonal in our representation. The parameter t of the unitary group $\hat{U}(t)$ that is generated by $-\hat{p}_t$ is easy to interpret: t represents the quantity that is conjugated to p_t in the classical theory and this is given by Eq. (58). Hence, $\hat{U}(t)$ describes the evolution of the shell states, for example, between the levels of the function $(U + V)/2$ on \mathcal{M} .

The missing piece of information of where the shell is on \mathcal{M} is carried by the quantity r of Eq. (58). We shall define the corresponding position operator in three steps.

First, we observe that r itself is not a Dirac observable, but the boost J is, and that the value of J at the surface $t = 0$ coincides with rp_t . It follows that the meaning of the Dirac observable Jp_t^{-1} is the position at the time $t = 0$. This is in a nice correspondence with the Newton-Wigner construction [40] on one hand, and with the notion of evolving constants of motion by Rovelli [27] on the other.

Second, we try to make Jp_t^{-1} into a symmetric operator on our Hilbert space. However, in the representation \mathcal{R}_1 , only \hat{J}^2 is meaningful. Let us then choose the following factor ordering for $J^2p_t^{-2}$:

$$\hat{r}^2 := \frac{1}{\sqrt{p}} \hat{J} \frac{1}{p} \hat{J} \frac{1}{\sqrt{p}} = -\sqrt{p} \frac{d^2}{dp^2} \frac{1}{\sqrt{p}}. \quad (60)$$

Other choices are possible; the above one makes \hat{r}^2 essentially a Laplacian and this simplifies the subsequent mathematics. Indeed, we can map \mathcal{K} unitarily to $L^2(\mathbb{R}_+)$ by sending each function $\psi(p) \in \mathcal{K}$ to $\tilde{\psi}(p) \in L^2(\mathbb{R}_+)$ as follows:

$$\tilde{\psi}(p) = \frac{1}{\sqrt{p}} \psi(p).$$

Then, the operator of squared position \tilde{r}^2 on $L^2(\mathbf{R}_+)$ corresponding to \hat{r}^2 is

$$\tilde{r}^2 \tilde{\psi}(p) = \frac{1}{\sqrt{p}} \hat{r}^2 \left(\sqrt{p} \tilde{\psi}(p) \right) = -\frac{d^2 \tilde{\psi}(p)}{dp^2} = -\tilde{\Delta} \tilde{\psi}(p).$$

Third, we have to extend the operator \hat{r}^2 to a self-adjoint one. The Laplacian on the half-axis possesses a one-dimensional family of such extensions [41]. The parameter is $\alpha \in [0, \pi)$ and the domain of $\tilde{\Delta}_\alpha$ is defined by the boundary condition at zero:

$$\tilde{\psi}(0) \sin \alpha + \tilde{\psi}'(0) \cos \alpha = 0.$$

The complete system of normalized eigenfunctions of $\tilde{\Delta}_\alpha$ can then easily be calculated:

$$\tilde{\psi}_\alpha(r, p) = \sqrt{\frac{2}{\pi}} \frac{r \cos \alpha \cos rp - \sin \alpha \sin rp}{\sqrt{r^2 \cos^2 \alpha + \sin^2 \alpha}};$$

if $\alpha \in (0, \pi/2)$, there is one additional bound state,

$$\tilde{\psi}_\alpha(b, p) = \frac{1}{\sqrt{2 \tan \alpha}} \exp(-p \tan \alpha),$$

so that

$$\begin{aligned} -\tilde{\Delta}_\alpha \tilde{\psi}_\alpha(r, p) &= r^2 \tilde{\psi}_\alpha(r, p), \\ -\tilde{\Delta}_\alpha \tilde{\psi}_\alpha(b, p) &= -\tan^2 \alpha \tilde{\psi}_\alpha(b, p). \end{aligned}$$

The corresponding eigenfunctions ψ_α of the operator \hat{r}_α^2 are:

$$\psi_\alpha(r, p) = \sqrt{\frac{2p}{\pi}} \frac{r \cos \alpha \cos rp - \sin \alpha \sin rp}{\sqrt{r^2 \cos^2 \alpha + \sin^2 \alpha}},$$

and we restrict ourselves to $\alpha \in [\pi/2, \pi]$, so that there are no bound states and the operator \hat{r} is self-adjoint. Indeed, \tilde{r} is the square root of $-\tilde{\Delta}_\alpha$, hence its eigenvalue for the bound state is imaginary.

To restrict the choice further, we apply the idea of Newton and Wigner [40]. First, the subgroup of Q_0 that preserves the surface $t = 0$ is to be found. This is, in our case, $U_D(s)$ generated by the dilatation D . Then, in the quantum theory, the eigenfunctions of the position at $t = 0$ are to transform properly under this group; this means that the eigenfunction for the eigenvalue r is to be transformed to that for the eigenvalue $e^{-s}r$, for each s and r . The dilatation group generated by \hat{D} acts on a wave function $\psi(p)$ as follows:

$$\psi(p) \mapsto U_D(s)\psi(p) = \psi(e^{-s}p),$$

where $U_D(s)$ is an element of the group parameterized by s . Applying $U_D(s)$ to $\psi_\alpha(r, p)$ yields

$$U_D(s)\psi_\alpha(r, p) = e^{-s/2} \sqrt{\frac{2p}{\pi}} \frac{r \cos \alpha \cos(e^{-s}rp) - \sin \alpha \sin(e^{-s}rp)}{\sqrt{r^2 \cos^2 \alpha + \sin^2 \alpha}}.$$

The factor $e^{-s/2}$ in the resulting functions of p keeps the system δ -normalized.

Let $\alpha = \pi/2$; then

$$U_D(s)\psi_{\pi/2}(r, p) = e^{-s/2}\psi_{\pi/2}(e^{-s}r, p).$$

Similarly, for $\alpha = \pi$,

$$U_D(s)\psi_{\pi}(r, p) = e^{-s/2}\psi_{\pi}(e^{-s}r, p),$$

but such relation can hold for no other α from the interval $[\pi/2, \pi]$, because of the form of the eigenfunction dependence on r . Now, Newton and Wigner require that

$$U_D(s)\psi(r, p) = e^{-s/2}\psi(e^{-s}r, p).$$

Then all values of α except for $\alpha = \pi/2$ and $\alpha = \pi$ are excluded.

We have, therefore, only two choices for the self-adjoint extension of \hat{r}^2 :

$$\psi(r, p) := \sqrt{\frac{2p}{\pi}} \sin rp, \quad r \geq 0, \quad (61)$$

and

$$\psi(r, p) := \sqrt{\frac{2p}{\pi}} \cos rp, \quad r \geq 0.$$

Let us select the first set, Eq. (61); by that, the construction of a position operator is finished. The construction contains a lot of choice: the large factor-ordering freedom, and the freedom of choosing the self-adjoint extension.

Another observable that we shall need is $\hat{\eta}$; this is to tell us the direction of motion of the shell at the time zero, having the eigenvalues $+1$ for all purely out-going shell states, and -1 for the in-going ones. In fact, in the classical theory, $\eta = -\text{sgn} p_r$, but p_r does not act as an operator on the Hilbert space \mathcal{K} , only p_r^2 , and the sign is lost.

Consider the classical dilatation generator $D = tp_t + rp_r$. It is a Dirac observable; at $t = 0$, its value is rp_r . Thus, for positive r , the sign of $-D$ at $t = 0$ has the required value. Hence, we have the relation:

$$\text{sgn} D = -\eta_{t=0}.$$

The eigenfunctions $\psi_a(p)$ of the operator \hat{D} are solutions of the differential equation:

$$\hat{D}\psi_a(p) = a\psi_a(p).$$

The corresponding normalized system is given by

$$\psi_a(p) = \frac{1}{\sqrt{2\pi}} p^{ia}.$$

Hence, the kernels $P_{\pm}(p, p')$ of the projectors \hat{P}_{\pm} on the purely out- or in-going states are:

$$P_+(p, p') = \int_{-\infty}^0 da \psi_a(p) \frac{\psi_a^*(p')}{p'}, \quad P_-(p, p') = \int_0^{\infty} da \psi_a(p) \frac{\psi_a^*(p')}{p'}$$

so that

$$(\hat{\eta}_0 \psi)(p) = \int_0^{\infty} dp' [P_+(p, p') - P_-(p, p')] \psi(p').$$

The observables \hat{p}_t , \hat{D} , \hat{r}_0 and $\hat{\eta}_0$ will suffice to work out a number of interesting predictions.

4.2 Motion of wave packets

We shall work with the family of wave packets on the energy half-axis that are defined by

$$\psi_{\kappa\lambda}(p) := \frac{(2\lambda)^{\kappa+1/2}}{\sqrt{(2\kappa)!}} p^{\kappa+1/2} e^{-\lambda p},$$

where κ is a positive integer and λ is a positive number with dimension of length. Using the formula

$$\int_0^{\infty} dp p^n e^{-\nu p} = \frac{n!}{\nu^{n+1}}, \tag{62}$$

which is valid for all non-negative integers n and for all complex ν that have a positive real part, we easily show that the wave packets are normalized,

$$\int_0^{\infty} \frac{dp}{p} \psi_{\kappa\lambda}^2(p) = 1.$$

Observe that $\psi_{\kappa\lambda}(p) = \psi_{\kappa 1}(\lambda p)$. The theory has been constructed so that it is scale invariant. The rescaling $p \mapsto \lambda p$ does not change the scalar product. The meaning of the parameter λ is, therefore, a scale: all energies concerning the packet will be proportional to λ^{-1} , all times and all radii to λ .

The expected energy,

$$\langle E \rangle_{\kappa\lambda} := \int_0^{\infty} \frac{dp}{p} p \psi_{\kappa\lambda}^2(p),$$

of the packet can be calculated by the formula (62) with the simple result

$$\langle E \rangle_{\kappa\lambda} = \frac{\kappa + 1/2}{\lambda}.$$

The (energy) width of the packet can be represented by the mean quadratic deviation (or dispersion),

$$\langle \Delta E \rangle_{\kappa\lambda} := \sqrt{\langle E^2 \rangle_{\kappa\lambda} - \langle E \rangle_{\kappa\lambda}^2},$$

which is

$$\langle \Delta E \rangle_{\kappa\lambda} = \frac{\sqrt{2\kappa + 1}}{2\lambda}.$$

In the Schroedinger picture, the time evolution of the packet is generated by $-\hat{p}_t$:

$$\psi_{\kappa\lambda}(t, p) = \psi_{\kappa\lambda}(p)e^{-ipt}.$$

Let us calculate the corresponding wave function $\Psi_{\kappa\lambda}(r, t)$ in the r -representation,

$$\Psi_{\kappa\lambda}(t, r) := \int_0^\infty \frac{dp}{p} \psi_{\kappa\lambda}(t, p) \psi(r, p),$$

where the functions $\psi(r, p)$ are defined by Eq. (61). Formula (62) then yields:

$$\Psi_{\kappa\lambda}(t, r) = \frac{1}{\sqrt{2\pi}} \frac{\kappa!(2\lambda)^{\kappa+1/2}}{\sqrt{(2\kappa)!}} \left[\frac{i}{(\lambda + it + ir)^{\kappa+1}} - \frac{i}{(\lambda + it - ir)^{\kappa+1}} \right]. \quad (63)$$

It follows immediately that

$$\lim_{r \rightarrow 0} |\Psi_{\kappa\lambda}(t, r)|^2 = 0.$$

The scalar product measure for the r -representation is just dr because the eigenfunctions (61) are normalized, so the probability to find the shell between r and $r + dr$ is $|\Psi_{\kappa\lambda}(t, r)|^2 dr$.

Our first important result is, therefore, that the wave packets start away from the center $r = 0$ and then are keeping away from it during the whole evolution. This can be interpreted as the *absence of singularity* in the quantum theory: no part of the packet is squeezed up to a point, unlike the shell in the classical theory.

Observe that the equation $\Psi_{\kappa\lambda}(t, 0) = 0$ is *not* a result of an additional boundary condition imposed on the wave function. It follows from the dynamics we have constructed. The nature of the question that we are studying requires that the wave packets start in the asymptotic region so that their wave function vanishes at $r = 0$ for $t \rightarrow -\infty$; this is the only condition put in by hand. The fact that the

dynamics preserves this equation is the property of the self-adjoint extensions of the Hamiltonian and the position operators.

Again, λ can be used to re-scale the radius, $r = \lambda\rho$ and the time, $t = \lambda\tau$ with the result

$$\Psi_{\kappa\lambda}(\lambda\tau, \lambda\rho) = \frac{1}{\sqrt{\lambda}} \Psi_{\kappa 1}(\tau, \rho) .$$

The factor $1/\sqrt{\lambda}$ is due to the scalar product in the r -representation being

$$(\Phi(r), \Psi(r)) = \int_0^\infty dr \Phi^*(r) \Psi(r) = \int_0^\infty d\rho \sqrt{\lambda} \Phi^*(\rho) \sqrt{\lambda} \Psi(\rho) ,$$

so that $\sqrt{\lambda}\Psi(\lambda\rho)$ is λ -independent.

A more tedious calculation is needed to obtain the time dependence $\langle r_t \rangle_{\kappa\lambda}$ of the expected radius of the shell,

$$\langle r_t \rangle_{\kappa\lambda} := \int_0^\infty dr r |\Psi_{\kappa\lambda}(t, r)|^2 . \quad (64)$$

The results that can be calculated analytically are (we skip the calculations; for details, see [34]):

$$\langle r_t \rangle_{\kappa\lambda} = \langle r_{-t} \rangle_{\kappa\lambda} \quad \forall \kappa, \lambda, t ,$$

that is all packets are time reversal symmetric. There is the minimal expected radius $\langle r_0 \rangle_{\kappa\lambda}$ at $t = 0$,

$$\langle r_0 \rangle_{\kappa\lambda} = \frac{1}{\pi} \frac{2^{2\kappa} (\kappa!)^2}{(2\kappa)!} \frac{\kappa + 1}{\kappa} \frac{\lambda}{\kappa + 1/2} > 0 . \quad (65)$$

Let us turn to the asymptotics $t \rightarrow \pm\infty$. We obtain for both cases $t \rightarrow \pm\infty$ and all λ and κ :

$$\langle r_t \rangle_{\kappa\lambda} \approx |t| + O(t^{-2\kappa}) . \quad (66)$$

We can also calculate the spread of the wave packet in r by means of the \hat{r} -dispersion $\langle \Delta r_t \rangle_{\kappa\lambda}$. The calculation of $\langle r^2 \rangle_{\kappa\lambda}$ is much easier than that of $\langle \hat{r} \rangle_{\kappa\lambda}$ because \hat{r}^2 is a differential operator in p -representation:

$$\langle r^2 \rangle_{\kappa\lambda} = -\lambda^2 \int_0^\infty \frac{dq}{q} \psi_{\kappa 1}^*(q, \tau) \left(-\sqrt{q} \frac{\partial^2}{\partial q^2} \frac{1}{\sqrt{q}} \psi_{\kappa 1}(q, \tau) \right) ,$$

where $q = \lambda p$. This gives

$$\langle r_t^2 \rangle_{\kappa\lambda} = t^2 + \frac{\lambda^2}{2\kappa + 1} . \quad (67)$$

For $\kappa \gg 1$, we can use the asymptotic expansion for the Γ -function to obtain

$$\frac{2^{2\kappa+1}(\kappa!)^2}{(2\kappa)!} = \sqrt{2\pi(2\kappa+1)} - \frac{1}{4}\sqrt{\frac{2\pi}{2\kappa+1}} + O(\kappa^{-3/2}) ;$$

that gives

$$\langle r_0 \rangle_{\kappa\lambda} = \lambda \left[\frac{1}{\sqrt{\pi\kappa}} + O(\kappa^{-3/2}) \right] . \quad (68)$$

Then, the spread of the packet for large κ is

$$\langle \Delta r_0 \rangle_{\kappa\lambda} = \lambda \left[\sqrt{\frac{\pi-2}{\kappa}} + O(\kappa^{-3/2}) \right] . \quad (69)$$

We observe that the spread is of the same order as the expected value. At $t \rightarrow \pm\infty$,

$$\langle \Delta r_{\pm\infty} \rangle_{\kappa\lambda} = \lambda \left[\frac{1}{\sqrt{2\kappa+1}} + O(t^{-2\kappa+1}) \right] . \quad (70)$$

Hence, the asymptotic spread is nearly equal to the spread at the minimum. This can be due to the light-like nature of the shell matter.

A further interesting question about the motion of the packets is about the portion of a given packet that moves in—is purely in-going—at a given time t . The portion is given by $\|\hat{P}_-\psi_{\kappa\lambda}\|^2$, where \hat{P}_- is the projector defined in Sec. 4.1. If we write out the projector kernel and make some simple rearrangements in the expression of the norm, we obtain:

$$\|\hat{P}_-\psi_{\kappa\lambda}\|^2 = \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dq'' \left(\int_0^{\infty} da \psi_a^*(e^{q'}) \psi_a(e^{q''}) \right) \psi_{\kappa\lambda}^*(t, e^{q''}) \psi_{\kappa\lambda}(t, e^{q'}),$$

where the transformation of integration variables p' and p'' to $e^{q'}$ and $e^{q''}$ in the projector kernels has been performed. The further calculation can be found in [34]. The results are:

$$\|\hat{P}_-\psi_{\kappa\lambda}\|_{t=0}^2 = 1/2 , \quad \|\hat{P}_-\psi_{\kappa\lambda}\|_{t \rightarrow -\infty}^2 = 1 , \quad \|\hat{P}_-\psi_{\kappa\lambda}\|_{t \rightarrow \infty}^2 = 0 .$$

Hence, we have one-to-one relation between in- and out-going states at the time of the bounce, while there are only in-going, or only out-going states at the infinity.

The obvious interpretation of these formulae is that quantum shell always bounces at the center and re-expands.

The result that the quantum shell bounces and re-expands is clearly at odds with the classical idea of black hole forming in the collapse and preventing anything that falls into it from re-emerging. It is, therefore, natural to ask, if the packet is squeezed enough so that an important part of it comes under its Schwarzschild radius. We

can try to answer this question by comparing the minimal expected radius $\langle r_0 \rangle_{\kappa\lambda}$ and its spread $\langle \Delta r_0 \rangle_{\kappa\lambda}$ with the expected Schwarzschild radius $\langle r_H \rangle_{\kappa\lambda}$ and its spread $\langle \Delta r_H \rangle_{\kappa\lambda}$ for the wave packet. For the Schwarzschild radius, we have

$$\langle r_H \rangle_{\kappa\lambda} = 2G\bar{M}_{\kappa\lambda} = (2\kappa + 1)\frac{L_P^2}{\lambda} ,$$

where L_P is the Planck length, and

$$\langle \Delta r_H \rangle_{\kappa\lambda} = 2G\langle \Delta M \rangle_{\kappa\lambda} = \sqrt{2\kappa + 1}\frac{L_P^2}{\lambda} .$$

We are to ask the question: for which λ and κ , the following inequality holds:

$$\langle r_0 \rangle_{\kappa\lambda} + \langle \Delta r_0 \rangle_{\kappa\lambda} < \langle r_H \rangle_{\kappa\lambda} - \langle \Delta r_H \rangle_{\kappa\lambda} .$$

If it holds, then the most of the packet becomes squeezed beyond the most of the Schwarzschild radius. From Eqs. (68) and (69), we obtain

$$\frac{\lambda^2}{L_P^2} \left(\frac{1}{\sqrt{\pi}} + \sqrt{\pi - 2} \right) < (2\kappa + 1)\sqrt{\kappa} - \sqrt{\kappa(2\kappa + 1)} .$$

Clearly, if $\lambda \approx L_P$, then the inequality holds for any large κ (starting from $\kappa = 2$). For larger λ , κ must be $(\lambda/L_P)^{4/3} \times$ larger. Hence, we must go to Planck regime in order that the packets cross their Schwarzschild radius.

To summarize: The packet can, in principle, fall under its Schwarzschild radius. Even in such a case, the packet bounces and re-expands.

This apparent paradox will be explained in the next section.

4.3 Grey horizons

In this section, we try to explain the apparently contradictory result that the quantum shell can cross its Schwarzschild radius in both directions. The first possible idea that comes to mind is simply to disregard everything that our model says about Planck regime. This may be justified, because the model can hardly be considered as adequate for this regime. Outside Planck regime, however, the shell bounces before it reaches its Schwarzschild radius and there is no paradox. However, the model *is* mathematically consistent, simple and solvable; it must, therefore, provide some mechanism to make the horizon leaky. We shall study this mechanism in the hope that it can work in more realistic situations, too.

To begin with, we have to recall that the Schwarzschild radius is the radius of a non-diverging null hyper-surface; anything moving to the future can cross such a hyper-surface only in one direction. The local geometry is that of an apparent horizon. (Whether or not an *event* horizon forms is another question; the answer to

it also depends on the geometry near the singularity [42]). However, as Einstein's equations are invariant under time reversal, there are two types of Schwarzschild radius: that associated with a black hole and that associated with a white hole. Let us call these Schwarzschild radii (or apparent horizons) themselves *black* and *white*. The explanation of the paradox that follows from our model is that quantum states can contain a linear combination of black and white (apparent) horizons, and that no event horizon ever forms. We call such a combination a *grey horizon*.

The existence of grey horizons can be shown as follows. The position and the “colour” of a Schwarzschild radius outside the shell is determined by the spacetime metric. For our model, this metric is a combination of purely gauge and purely dependent degrees of freedom, and so it is determined, within the classical version of the theory, by the physical degrees of freedom through the constraints.

The metric can be calculated along a Cauchy surface Σ that intersect the shell with total energy E , direction of radial motion η , and the radius r . The result of this calculation (see [34]) is: If the shell is contracting, $\eta = -1$, then any space-like surface containing such a shell can at most intersect an *out-going* apparent horizon at the radius $R = 2GE$. Analogous result holds for $\eta = +1$, where the shell is expanding: the apparent horizon is then in-going. The horizon radius is determined by the equation $R_H = 2GE$, and the horizon will cut Σ if and only if $R_H > r$. We can assign the value $+1$ (-1) to the horizon that is out- (in-)going and denote the quantity by c (colour: black or white hole). To summarize:

1. The condition that an apparent horizon intersects Σ is $r < 2GE$.
2. The position of the horizon at Σ is $R_H = 2GE$.
3. The colour c of this horizon is $c = \eta$.

In this way, questions about the existence and colour of an apparent horizon outside the shell are reduced to equations containing dynamical variables of the shell. In particular, the result that $c = \eta$ can be expressed by saying that the shell always creates a horizon outside that cannot block its motion. All that matters is that the shell can bounce at the singularity (which it cannot within the classical theory).

These results can be carried over to quantum mechanics after quantities such as $2GE - r$ and η are expressed in terms of the operators describing the shell. Then we obtain a “quantum horizon” with the “expected radius” $2G\langle E \rangle$ and with the “expected colour” $\langle \eta \rangle$ to be mostly black at the time when the expected radius of the shell crosses the horizons inwards, neutrally grey at the time of the bounce and mostly white when the shell crosses it outwards.

This proof has, however, two weak points. First, the spacetime metric on the background manifold is not a gauge invariant quantity; although all gauge invariant

geometrical properties can be extracted from it within the classical version of the theory, this does not seem to be possible in the quantum theory [43]. Second, calculating the quantum spacetime geometry along hyper-surfaces of a foliation on a given background manifold is foliation dependent. For example, one can easily imagine two hyper-surfaces Σ and Σ' belonging to different foliations, that intersect each other at a sphere outside the shell and such that Σ intersects the shell in its in-going and Σ' in its out-going state. Observe that the need for a foliation is only due to our insistence on calculating the quantum metric on the background manifold.

The essence of these problems is the gauge dependence of the results of the calculation. However, it seems that this dependence concerns only details such as the distribution of different hues of grey along the horizon, not the qualitative fact that the horizon exists and changes colour from almost black to almost white. Still, a more reliable method to establish the existence and properties of grey horizons might require another material system to be coupled to our model; this could probe the spacetime geometry around the shell in a gauge-invariant way.

4.4 Concluding remarks

Comparison of the motion of wave packets of Sec. 4.2 with the classical dynamics of the shell as described in Sec. 3.1 shows a notable difference. Whereas all classical shells cross their Schwarzschild radius and reach the singularity in some stage of their evolution, the quantum wave packets never reach the singularity, but always bounce and re-expand; some of them even manage to cross their Schwarzschild radius during their motion. This behaviour is far from being a small perturbation around a classical solution if the classical spacetime is considered as a whole. Even locally, the semi-classical approximation is not valid near the bouncing point. It is surely valid in the whole asymptotic region, where narrow wave packets follow more or less the classical trajectories of the shell.

The most important question, however, concerns the validity of the semi-classical approximation near the Schwarzschild radius. We have seen that the geometry near the radius can resemble the classical black hole geometry in the neighbourhood of the point where the shell is crossing the Schwarzschild radius inwards. Then, the radius changes its colour gradually and the geometry becomes very different from the classical one. Finally, near the point where the shell crosses the Schwarzschild radius outwards, the radius is predominantly white and the quantum geometry can be again similar to the classical geometry, this time of a white hole horizon.

If the change of colour is very slow then the neighbourhood of the inward crossing where the classical geometry is a good approximation can be large. It seems that sufficiently large scattering time would allow for arbitrarily slow change of colour. We cannot exclude, therefore, that the quantum spacetime contains an extended

region with the geometry resembling its classical counterpart near a black hole horizon. This can be true even if the quantum spacetime as a whole differs strongly from any typical classical collapse solution.

One can even imagine the following scenario (which needs a more realistic model than a single thin shell). A quantum system with a large energy collapses and re-expands after a huge scattering time. The black hole horizon phase is so long, that Hawking evaporation becomes significant and must be taken into account in the calculation. It does then influence the scattering time and the period of validity of the black hole approximation. The black hole becomes very small and only then the change of horizon colour becomes significant. The white hole stage is quite short and it is only the small remnant of the system that, finally, re-expands. The whole process can still preserve unitarity. In fact, this is a scenario for the issue of Hawking evaporation process. It is not excluded by the results of the present paper.

One can also consider the following conception. At and under the Schwarzschild radius, the local spacetime geometry for a white Schwarzschild radius is very, and measurably, different from that of a black one. However, outside the Schwarzschild radius, the local geometries of both cases are isometric to each other so that the isometry need not contain the time reversal. Hence, the quantum geometry outside a very grey horizon need not actually differ from the classical geometry around a black hole (locally) very much.

All these speculations must be investigated and made more precise. If an observer staying at the radius R_B measures the proper time $\Delta T(R_B)$ between the shell wave packet crossing his position in and out, then of course $\lim_{R_B \rightarrow \infty} \Delta T(R_B) = \infty$. In the collision theory, one considers, therefore, some standard collision with $\Delta_s T(R_B)$ and takes the limit only after the subtraction,

$$\lim_{R_B \rightarrow \infty} (\Delta T(R_B) - \Delta_s T(R_B)) .$$

This limit, if finite, is called *time delay* [44]. A particularly difficult example is Coulomb scattering, for which the divergence in $\Delta T(R_B)$ involves not only linear, but also logarithmic terms. This is due to the long range of Coulomb potential and is, therefore, called *infrared divergence*. One can still find a subtraction for the Coulomb case because the logarithmic term depends only on charges and so has the same leading part for a whole class of scattering processes.

In our case, there are also logarithmic terms, but they depend on the energy of the shell (energy is the charge for gravitation). No reasonable subtraction seems, therefore, possible. One had better work at a finite radius R_B all time. To calculate our $\Delta T(R_B)$, one has to use the quantum geometry outside the shell.

But what is the quantum geometry? In the classical version of general relativity, a metric tensor field written in particular coordinates contains all information about

local geometric properties. A change in the coordinates does not lead to any change of these properties because they can be calculated from the transformed metric. In a quantum theory, however, coordinate transformations is a delicate issue. Let us turn to our simple system to see why. We have chosen the coordinates U and V that are uniquely determined for each solution by means of their geometric properties. The components $A(U, V)$ and $R(U, V)$ of the four-metric with respect to these coordinates are geometric quantities themselves. Nobody wants to deny that the quantities $A(\eta, M, w; U, V)$ and $R(\eta, M, w; U, V)$ are Dirac observables for each fixed value of U and V : they are then just functions of η , M and w , which *are* Dirac observables. In the quantum theory, one can try to promote these quantities to operators by replacing η , M and w by the corresponding quantum operators and by choosing a suitable factor ordering. Let us suppose that the quantities $\hat{A}(U, V)$ and $\hat{R}(U, V)$ obtained in this way are well defined operators for each value of U and V . Cannot we calculate all geometric properties near each point (U, V) from these operators?

Problems emerge if we choose to work with a different set of geometric coordinates. Let us, for example, pass to the Schwarzschild coordinates T and R ; near infinity, they are a very natural choice. Again, we can work out the form of the metric components $g_{00}(\eta, M, w; T, R)$ and $g_{11}(\eta, M, w; T, R)$ for each solution and try to construct the operators $\hat{g}_{00}(T, R)$ and $\hat{g}_{11}(T, R)$ from them. Is there any transformation within the quantum theory analogous to the classical coordinate transformation and providing a basis for a proof that the local geometric properties calculated from $\hat{A}(U, V)$ and $\hat{R}(U, V)$ are the same as those calculated from $\hat{g}_{00}(T, R)$ and $\hat{g}_{11}(T, R)$? A problem is that the transformation between $\{U, V\}$ and $\{T, R\}$ is field dependent. Indeed, for $\eta = +1$, it is given by Eqs. (26) and (27). The right-hand sides depend not only on T_∞ (which coincides with our T here) and R , but also on M and w (the full transformation depends also on η). Hence, if we assume that the quantities T and R are just parameters, then the quantities U and V are operators and vice versa. However, the quantities T and R are parameters in the operators $\hat{g}_{00}(T, R)$ and $\hat{g}_{11}(T, R)$ and the quantities U and V are parameters in $\hat{A}(U, V)$ and $\hat{R}(U, V)$.

It seems, therefore, that there is a problem of principle in addition to the factor-ordering problem to transform the operators $\hat{A}(U, V)$ and $\hat{R}(U, V)$ to $\hat{g}_{00}(T, R)$ and $\hat{g}_{11}(T, R)$. The conclusion is that quantum geometry cannot be described in a coherent manner analogous to the differential geometry, see also [43].

At the present stage, we try to calculate each geometric property for itself. Thus, in his PhD thesis, M. Ambrus, to whom I owe all my knowledge about collision theory, tries to define and calculate the scattering times. Another attempt is to define quantum geometry in a similar way as the classical geometry is defined: by properties of test particles. This might work even near the Schwarzschild and zero

radius. Thus, we are trying with I. Kouletsis, to quantize an analogous system containing *two* shells and to use the second shell as a spy probing the quantum geometry created by the first one. For some preliminary results see [45], [28] and [46].

The calculations of this paper are valid only for null shells. Similar calculations for massive shells have been performed in [47]. There has been re-expansion and unitarity for massive shells if the rest mass has been smaller than the Planck mass (10^{-5} g). It is very plausible that the interpretation of these results is similar to that given in the present paper. Thus, we can expect the results valid at least for all “light” shells. There is, in any case, a long way to any astrophysically significant system and a lot of work is to be done before we can claim some understanding of the collapse problem.

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